Bayesian source localization with uncertain Green’s function in an uncertain shallow water ocean

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Abstract

Matched-field acoustic source localization is a challenging task when environmental properties of the oceanic waveguide are not precisely known. Errors in the assumed environment (mismatch) can cause severe degradations in localization performance. This paper develops a Bayesian approach to improve robustness to environmental mismatch by considering the waveguide Green’s function to be an uncertain random vector whose probability density accounts for environmental uncertainty. The posterior probability density is integrated over the Green’s function probability density to obtain a joint marginal probability distribution for source range and depth, accounting for environmental uncertainty and quantifying localization uncertainty. Because brute-force integration in high dimensions can be costly, an efficient method is developed in which the multi-dimensional Green’s function integration is approximated by one-dimensional integration over a suitably defined correlation measure. An approach to approximate the Green’s function covariance matrix, which represents the environmental mismatch, is developed based on modal analysis. Examples are presented to illustrate the method and Monte-Carlo simulations are carried out to evaluate its performance relative to other methods. The proposed method gives efficient, reliable source localization and uncertainties with improved robustness toward environmental mismatch.

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I Introduction

Matched-field processing (MFP) is an appealing technique for source localization in underwater acoustics [1]. MFP takes full advantage of the oceanic waveguide propagation features to infer the source position by comparing data measured on an array of spatially distributed sensors with replicas of the acoustic field derived from the wave equation and a model of the oceanic waveguide [2, 3]. However, mismatch between the assumed oceanic waveguide and the true environment (e.g. errors in bathymetry, bottom properties or sound speed profile) may result in poor localization performance. Mismatch introduces biased estimates at high signal-to noise ratio (SNR) [4–6] and degrades performance with respect to noise [7]. Since it is unlikely to have accurate knowledge of the oceanic waveguide environment in realistic applications of MFP, mismatch is a serious problem. Several methods have been put forward to tackle this problem and make source localization robust to uncertainty in the environment.

The matched-mode processing (MMP) approach is an alternative to MFP that explicitly uses the modal description of the acoustic field to estimate source position. MMP methods first require the decomposition of the acoustic field into its propagating modes and then compare the measured modes to simulated mode replicas to infer the desired parameter values [8–10]. Working in mode space provides physical insight into the estimation scheme and robustness is achieved by retaining only the modes that are less affected by environmental mismatch [10–12]. However, modal decomposition is a critical step which is often inaccurate, particularly when the array does not provide an adequate spatial sampling of the acoustic field, and errors in modal amplitudes may considerably reduce MMP localization performance [10, 11, 13]. To overcome this issue, Tabrikian et al. [14] derived a maximum likelihood (ML) MFP estimator, considering the acoustic field as the sum of two kinds of modes: predictable modes weakly affected by the environment and unpredictable modes
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strongly affected by the environment and regarded as nuisance parameters. This estimator showed significantly better performance than standard MMP in the presence of mismatch. Extending this idea, Liu et al. [15] recently derived another ML MFP estimator that was found to be slightly more robust. These mode-based approaches address the mismatch problem by ignoring the information carried by the parts of the acoustic field that are the most sensitive to the oceanic waveguide environment and therefore imply a trade off between the loss of information on source location and sensitivity to mismatch.

Other approaches incorporate uncertain environmental parameters as additional unknowns constrained by a priori information in the localization problem [16–24], which makes the localization robust to mismatch with respect to these parameters. Two general approaches, namely focalization and marginalization, have been considered. The focalization approach consists of finding the globally optimal solution jointly for the localization and environmental parameters [16–18]. The marginalization approach formulates the problem within a Bayesian framework [19–24]. Bayes’ rule relates the a priori probability density and the likelihood to the posterior probability density (PPD) which quantifies the probability of the unknown parameters given the measured data and prior information. The analysis is carried out for the extended PPD involving the localization parameters as well as the environmental parameters. Integration (marginalization) of the multidimensional PPD over the environmental parameters gives the joint marginal probability distribution over source range and depth. Hence, marginalization can provide a robust estimate of the source position as well as a quantitative measure of localization uncertainty which may be of critical importance in an operational context for decision making. Focalization and marginalization rely on a parameterization of the waveguide and are computationally expensive since they require numerical propagation modeling for a large number of realizations of the unknown environmental parameters.
In this paper we propose a novel Bayesian approach to address the mismatch problem with no unknown environmental parameters added to the localization problem and no requirement for repeated propagation modeling. The idea is to relax the assumption that the Green’s function of the supposed oceanic waveguide is a deterministic quantity. Instead, the Green’s function is considered as a random vector that accounts for environmental uncertainty through its probability density. The analysis is carried out for a PPD involving the localization parameters as well as the uncertain Green’s function. Integrating (marginalizing) the PPD over the uncertain Green’s function provides the joint marginal probability distribution over source range and depth, quantifying the uncertainties in localization. This integration does not require repeated propagation modeling. Furthermore, it is reduced to a one-dimensional integral which greatly increases computational efficiency. The impact of environmental uncertainty on the Green’s function is handled by making some general assumptions regarding the effect of mismatch on modal propagation. This approach has the additional advantage that the uncertainty is not limited to mismatch in specifically-chosen environmental parameters, but applies generally to all factors influencing propagation (e.g., waveguide structure, range-dependent effects). The ability of the method to give reliable marginal PPDs for source location is illustrated through simulated examples, and the performance of the method is assessed by Monte-Carlo simulations.

The remainder of this paper is organized as follows. Section II presents the Bayesian formulation of the MFP inverse problem. Section III describes our Bayesian MFP approach including the marginalization technique and provides a discussion on the Green’s function model for uncertain ocean environments. Section IV presents the results of the proposed method through simulated examples and Monte-Carlo performance analyses. Finally, Section V summarizes and discusses the results of this work.
**Notation:** Throughout this paper, lowercase boldface letters denote vectors, e.g., \( \mathbf{x} \), and uppercase boldface letters denote matrices, e.g., \( \mathbf{A} \). The superscripts \( ^T \) and \( ^H \) mean transposition and Hermitian transposition, respectively. The trace operator is denoted \( \text{tr}(\cdot) \). We use \( \text{diag}(\mathbf{x}) \) to designate a diagonal square matrix whose main diagonal contains the elements of vector \( \mathbf{x} \). The \( N \times N \) identity matrix is denoted by \( \mathbf{I}_N \). The operator \( |\cdot| \) means determinant for matrices and modulus for scalars. The operator \( \|\cdot\| \) designates the standard Euclidean norm. The distribution of a jointly proper complex Gaussian random vector with mean \( \mathbf{m} \) and covariance matrix \( \mathbf{R} \) is denoted \( \mathcal{CN}(\mathbf{m}, \mathbf{R}) \). Finally, \( E[\cdot] \) denotes expectation, \( P_r(\cdot) \) denotes probability, and \( p(\cdot) \) is a probability density function.

**II Bayesian matched-field processing**

**A Bayesian inference**

In a Bayesian framework, inference about an unknown parameter set \( \mathbf{\theta} \) is made through the analysis of the conditional probability density function \( p(\mathbf{\theta}|\mathbf{y}) \) of \( \mathbf{\theta} \) given the measured data \( \mathbf{y} \). This quantity, referred to as the PPD, is obtained using Bayes’ rule which can be expressed as

\[
p(\mathbf{\theta}|\mathbf{y}) \propto p(\mathbf{y}|\mathbf{\theta})p(\mathbf{\theta}),
\]

(1)

where \( p(\mathbf{\theta}) \) is the prior information about \( \mathbf{\theta} \) and \( p(\mathbf{y}|\mathbf{\theta}) \) is the probability of obtaining the measured data \( \mathbf{y} \) given a realization of the parameter set \( \mathbf{\theta} \). The distribution \( p(\mathbf{y}|\mathbf{\theta}) \) is defined as the likelihood function when interpreted as a function of \( \mathbf{\theta} \). The PPD represents the state of knowledge of \( \mathbf{\theta} \) incorporating both prior information and data information. Therefore, it provides all available information to make inferences on the unknown parameters. The PPD can be used to estimate the parameters through several Bayesian estimation methods such as the maximum \textit{a posteriori}
(MAP) estimator or the posterior mean estimator. The MAP estimator is defined as

$$\hat{\theta} = \arg \max_{\theta} p(\theta|y),$$  
(2)

and the posterior mean is defined as

$$\bar{\theta} = \int \theta p(\theta|y) \, d\theta.$$  
(3)

The MAP has the minimum probability of error when the parameters have discrete probability distributions whereas the posterior mean is the minimum mean-squared error estimator [25]. Further analysis of the PPD can provide indications of the reliability of the estimate and of other possible solutions to the inverse problem.

B Matched-field processing likelihood function

The likelihood function is determined by the statistical distribution of the data. In MFP, the data are the output of an $N$-element array on which impinge the waveforms from an acoustic source located some distance away in the ocean waveguide. For each time segment (snapshot) $l$ and each frequency $f_m$, the complex (frequency domain) array output is modeled by the $N \times 1$ vector

$$y_l(f_m) = s_l(f_m)g(f_m, \theta) + w_l(f_m), \quad l = 1, ..., L, \ m = 1, ..., M,$$  
(4)

where

- $L$ is the number of snapshots available for each of the $M$ frequencies,
- $g(f_m, \theta)$ is a complex $N \times 1$ vector representing the transfer function or Green’s function of the waveguide at frequency $f_m$ for propagation from the source to each of the $N$ receivers,
- $s_l(f_m)$ is a complex scalar representing the source amplitude and phase, and
- \( \mathbf{w}_l(f_m) \) is a complex \( N \times 1 \) vector representing circularly-symmetric, zero-mean Gaussian noise that is independent of the source signal.

The noise is assumed to be white in frequency and time with positive definite spatial covariance matrix \( \Sigma_w(f_m) = \sigma_w^2(f_m) \mathbf{C}_w(f_m) \), where \( \sigma_w^2(f_m) \) is the noise variance and \( \mathbf{C}_w(f_m) \) is the noise spatial correlation matrix over the array. This noise model is suited to ambient noise which is always present in an operational context. Its interpretation is sometimes broadened to include modeling errors such as environmental mismatch [26, 27]. However, since modeling errors are generally not temporally-independent and additive, this interpretation may be problematic particularly when these errors are large. In this paper, environmental mismatch is addressed in terms of an uncertain Green’s function, and is not considered a component of the noise model. Since the matrices \( \mathbf{C}_w(f_m) \) are positive definite, their square root decompositions are invertible and can be obtained via the Cholesky decomposition \( \mathbf{C}_w(f_m) = \mathbf{L}_w(f_m) \mathbf{L}_w^H(f_m) \). Therefore, the whitening transform

\[
\tilde{y}_l(f_m) = \mathbf{L}_w^{-1}(f_m)\mathbf{y}_l(f_m),
\]
\[
\tilde{g}(f_m, \theta) = \mathbf{L}_w^{-1}(f_m)\mathbf{g}(f_m, \theta), \quad l = 1, \ldots, L, \quad m = 1, \ldots, M
\] (5)

can be applied if the matrix \( \mathbf{C}_w(f_m) \) is known, which allows reduction to the white noise case.

In the remainder of this paper we will work in whitened coordinates and assume that \( \Sigma_w(f_m) = \sigma_w^2(f_m) \mathbf{I}_N \) to simplify subsequent derivations and analyses. It should be noted that characterizing the ambient noise field by a spatial covariance matrix may not provide a precise representation in all cases, and that accurately estimating the correlation matrix \( \mathbf{C}_w(f_m) \) can be a challenging task. Errors between the assumed noise statistics and the true statistics may hinder the estimation of \( \theta \), particularly when SNR is low. This problem has been considered in other work [28–31], and
is not considered further in the present paper which focuses on localization with environmental mismatch.

In the original development of Bayesian MFP by Richardson and Nolte [19], the source signal was regarded as an unknown quantity \( s_t(f_m) = s_l(f_m) \) with a Rayleigh distributed amplitude and a uniform phase. However, a deterministic model which considers the source term \( s_t(f_m) \) as a deterministic quantity is more flexible and has been used in many recent applications (e.g., [22–24, 26, 27, 32–34]). In what follows we use this deterministic source signal model. Thus, the likelihood can be expressed as

\[
p(y|\theta) = \frac{1}{\prod_{m=1}^{M} \prod_{l=1}^{L} |\pi \sigma_w^2(f_m)|^N} \prod_{m=1}^{M} \prod_{l=1}^{L} \exp \left( -\frac{||y_l(f_m) - s_l(f_m)g(f_m, \theta)||^2}{\sigma_w^2(f_m)} \right). \tag{6}
\]

The source terms \( s_l(f_m) \) and the noise variances \( \sigma_w^2(f_m) \) are usually not known in realistic applications of MFP and must be treated as nuisance parameter in the Bayesian inversion process. A few techniques have been investigated to deal with these nuisance parameters [26, 27]. A convenient and reliable technique involves replacing the source terms and/or the noise variance by their maximum likelihood estimate (which are functions of the model parameters \( \theta \) ) [22–24, 26, 27, 32–34]. When the source terms are unknown but the noise variances are known, this procedure leads to [33, Eq. (3.3)]

\[
p(y|\theta) \propto \prod_{m=1}^{M} (\pi \sigma_w^2(f_m))^{-LN} \exp \left( -L \frac{\phi_m(\theta)}{\sigma_w^2(f_m)} \right), \tag{7}
\]

and when both the source terms and the noise variances are unknown we get [33, Eq. (3.11)]

\[
p(y|\theta) \propto \prod_{m=1}^{M} \exp (-LN \log_e \phi_m(\theta)), \tag{8}
\]

where

\[
\phi_m(\theta) = \operatorname{tr}(R_m) - \frac{g^H(f_m, \theta)R_mg(f_m, \theta)}{g^H(f_m, \theta)g(f_m, \theta)}, \quad R_m = \frac{1}{L} \sum_{l=1}^{L} y_l(f_m)y_l^H(f_m). \tag{9}
\]
One of these two likelihood functions can be used depending upon the information available on the noise variance. Note that $\phi_m$ is based on the linear Bartlett correlator between measured and modeled fields, as commonly applied in matched-field localization.

### III Bayesian source localization with uncertain Green’s function

For source localization, the unknown parameter set $\theta$ represents the range and depth of the source. In a Bayesian framework, uncertainty in the environment is generally handled by adding environmental parameters $\theta_{env}$ to $\theta$ [19–24]. The PPD over the source position parameter $\theta$ is then obtained by integrating the full PPD over the environmental parameters:

$$p(\theta|y) = \int p(\theta, \theta_{env}|y) d\theta_{env}. \quad (10)$$

This approach, known as marginalization, is attractive; however, it is also computationally intensive because it requires repeatedly simulating the acoustic field for the enlarged set of environmental parameter sampled during the integration. In this paper, no environmental parameters are added to $\theta$. Instead, environmental uncertainty is addressed by considering its approximate effect on the Green’s function. In particular, the Green’s function is considered as a random vector whose probability density accounts for environmental uncertainty. The marginal PPD over range and depth is obtained by integrating the full PPD over the uncertain Green’s function. The integral is reduced to a one-dimensional integral which greatly speeds up the integration and makes the whole process computationally efficient (much more efficient than focalization or marginalization over specific environmental parameters).

This section presents this approach to MFP source localization. First, the theory of Bayesian source localization with an uncertain Green’s function is presented in Sec. III A. The proposed
methodology requires a computationally efficient integration technique over the Green’s function probability density and a probabilistic model of the Green’s function uncertainty. The efficient integration technique is described in Sec. III B and Green’s function models for uncertain environment are discussed in Sec. III C. For simplicity, only single-frequency MFP is presented in what follows. The results can easily be extended to the multi-frequency problem since the multi-frequency likelihood is the product of the single-frequency counterparts under the common assumption of uncorrelated errors over frequency.

A Theory

It is important to note here that the Green’s function is taken to be constant in time assuming the environment does not vary significantly over the time of the $L$ snapshots considered. The unknown parameter set $\theta$ solely represents the localization parameters. Suppose that the Green’s function of the assumed waveguide is a random vector with a prior density function $p(g|\theta)$ that accounts for environmental uncertainty; the PPD of the localization parameters can then be obtained through marginalization over $p(g|\theta)$

$$p(\theta|y) \propto \int p(y|g, \theta)p(g|\theta)dg \ p(\theta).$$  \hspace{1cm} (11)

In what follows, the dependence of the Green’s function $g$ on $\theta$ will be omitted for clarity.

The prior $p(g)$ aims at characterizing the a priori knowledge of the possible mismatch by its effect on the Green’s function of the waveguide. It represents the state of belief on the mismatch. When the environment is uncertain, the Green’s function is no longer a deterministic vector but a random vector which could, in principle, lie anywhere in the $N$-dimensional space. However, in practice, the Green’s function generally retains some structure and lies preferentially in some regions of this space. A simple and flexible way to characterize this is to consider the Green’s func-
tion as a zero-mean Gaussian random vector characterized by a positive definite spatial covariance matrix $\Sigma_g$, i.e. $g \sim \mathcal{CN}(0, \Sigma_g)$. This spatial covariance matrix parameterizes the statistical dispersion of the Green’s function, and reflects our physical knowledge of the propagation (see Fig. 1 for a schematic illustration of this concept). Note that as we are considering unknown source terms, only the directivity of the Green’s function is important and not its magnitude (see Eq. (9)).

Figure 1: (Color online) Schematic illustration of the Green’s function model in $N = 3$ dimensions and a real vector space. The axes $e_1$, $e_2$, $e_3$ represent the standard basis. The axes $u_1$, $u_2$, $u_3$ represent the eigenvectors of $\Sigma_g$ and the crosses represent normalized Green’s function generated from $\Sigma_g$. When the environment is uncertain, the Green’s function is no longer a deterministic vector but a random vector that may lie anywhere in the $N$ dimensional space. However, it generally retains some structure and lies preferentially in some regions of this space. In this schematic illustration, the Green’s functions are mostly oriented toward $u_1$ but may also direct toward $u_2$ while they direct very little toward $u_3$. This behavior is defined by the covariance matrix $\Sigma_g = U \Lambda U^H$, where $U = [u_1, u_2, u_3]$, $\Lambda = \diag(\sigma_1^2, \sigma_2^2, \sigma_3^2)$ and $\sigma_1 > \sigma_2 \gg \sigma_3$.

The description of environmental uncertainty within the covariance matrix $\Sigma_g$ is discussed later
in Sec. III C. The following subsection describes the marginalization technique used to evaluate the integral in Eq. (11) assuming that the covariance matrix $\Sigma_g$ is available.

B Green’s function marginalization

The marginalization of the Green’s function over the PPD involves an $N$-dimensional integral since $g$ is a $N \times 1$ vector. Monte-Carlo integration of such a high-dimensional integral is prohibitively costly in general and more efficient techniques are required. The idea here is to reduce this $N$-dimensional integral to a one-dimensional integral. Denote $u$ the scalar quantity

$$u = \frac{1}{\text{tr}(R)} g^H R g,$$

(12)

with $R$ defined in Eq. (9). The likelihoods in Eqs. (7) and (8) can be expressed as functions of $u$ instead of $g$ since $\phi$ in Eq. (9) can be written as $\phi = \text{tr}(R)(1 - u)$. The PPD of the localization parameters, Eq. (11), can therefore be obtained without any approximation through

$$p(\theta|y) \propto \int_0^1 p(y|u)p(u)du \; p(\theta),$$

(13)

as $u$ is a random variable in the interval $[0, 1]$ (The Cauchy-Schwarz inequality applies to the positive semi-definite matrices $R$ and $gg^H$ with the trace operator as the inner product which leads to a proof that $u \leq 1$). Note that $u$ is similar to a squared correlation between the observed data covariance matrix and the Green’s function.

The integration in Eq. (13) requires the evaluation of $p(u)$. The variable $u$ is a ratio of quadratic forms in the Gaussian-distributed Green’s function vector $g$ with covariance matrix $\Sigma_g$. Its probability density $p(u)$ is not known analytically but can be computed using results on quadratic forms in Gaussian variables [35]. The cumulative distribution function (CDF) $F_U(u)$ of $u$ can be written
as:

\[ F_U(u) = P_r \left( \frac{1}{\text{tr}(R)} \frac{g^H R g}{g^H g} < u \right), \]  

(14)

\[ = P_r \left( g^H \left[ \frac{R}{\text{tr}(R)} - u I_N \right] g < 0 \right). \]  

(15)

Let \( h \) be defined such that \( g = \Sigma_g^{1/2} h \). This operation is actually a whitening transform and \( h \sim \mathcal{C}\mathcal{N}(0, I_N) \). The CDF \( F_U(u) \) can then be expressed as

\[ F_U(u) = P_r \left( h^H A_u h < 0 \right), \]  

(16)

where

\[ A_u = \Sigma_u^{1/2} \left[ \frac{R}{\text{tr}(R)} - u I_N \right] \Sigma_u^{1/2}. \]  

(17)

Therefore, \( F_U(u) \) is equivalent to the CDF at zero of the quadratic form \( h^H A_u h \). Let \( \{\lambda_{u,i}, i = 1, \ldots, P \} \) be the positive eigenvalues of the matrix \( A_u \) and assume that none of them are repeated; the CDF can then be computed as [35]

\[ F_U(u) = 1 - \sum_{i=1}^{P} \frac{\lambda_{u,i}^{N-1}}{\prod_{l \neq i} (\lambda_{u,i} - \lambda_{u,l})}, \]  

(18)

where \( P \) is the number of positive eigenvalues of \( A_u \). None of the positive eigenvalues of the matrix \( \left[ \frac{R}{\text{tr}(R)} - u I_N \right] \) are repeated, but we are unaware if this is sufficient to ensure that none of the positive eigenvalues of \( A_u \) are repeated. However, this seems logical and has always been true for all cases we have considered. In any case, a more complex expression may be used if some positive eigenvalues are repeated [35]. The probability density \( p(u) \) can be calculated by the numerical differentiation of \( F_U(u) \), and the integrand in Eq. (13) can therefore be evaluated at a finite set of points to carry out the numerical integration. In practice, integration on a regular grid with 100 sampling points has been shown to provide good results.
This subsection introduced the marginalization technique over the uncertain random Green’s function assuming that its covariance matrix \( \Sigma_g \) is available. The next subsection presents an approach to approximate this covariance matrix.

C Green’s function model for environmental uncertainty

The covariance matrix \( \Sigma_g \) characterizes the uncertainty about the Green’s function of the waveguide due to uncertainty in the oceanic waveguide environment. The method presented above applies for any positive definite matrix \( \Sigma_g \) and many approaches may be considered to design such a matrix. For example, the matrix \( \Sigma_g \) could be simply estimated once and for all by computing some Monte-Carlo realizations of the Green’s function from realizations of environmental parameters [20, 36]. However, uncertainty in the environment is addressed here through an uncertain Green’s function, which can more generally represent uncertainty on propagation as a whole. In this paper, we introduce some assumptions regarding the effect of mismatch on modal propagation that allow the construction of a matrix \( \Sigma_g \) that approximately characterizes the Green’s function uncertainty. This approach has the advantage that the uncertainty is not limited to mismatch in specifically-chosen parameters, but may apply generally to all factors influencing propagation (e.g., waveguide structure, range-dependent effects).

The assumptions on the effect of mismatch on modal propagation are first introduced, and then the methodology to construct the matrix \( \Sigma_g \) using these assumptions is presented.

1 Impact of environmental uncertainty on acoustic propagation

In shallow water, acoustic propagation can be described by normal mode theory [37]. For an acoustic source at depth \( z_s \) in a shallow-water waveguide, the transfer function of the waveguide
after propagation over a range \( r_s \) for a receiver at depth \( z_r \) is a sum of modes given by [37]:

\[
h(f, r_s, z_s, z_r) = Q \sum_{m=1}^{M} \psi_m(f, z_s) \psi_m(f, z_r) \frac{e^{jk_{rm}(f) r_s}}{\sqrt{k_{rm}(f) r_s}}. \tag{19}
\]

where \( k_{rm} \) and \( \psi_m \) are the horizontal wavenumber and modal depth function of mode \( m \), and \( M \) is the number of propagating modes. The quantity \( Q = \frac{e^{-j\pi/4}}{\sqrt{8\pi \rho(z_s)}} \) represents a constant factor with \( \rho(z_s) \) as the water density at the source depth. The horizontal wavenumbers \( k_{rm} \) and modal depth functions \( \psi_m \) depend on the oceanic waveguide environment (bathymetry, geoacoustic properties of the bottom, and water column sound speed).

In practice, different kinds of modes experience different kinds of propagation in a shallow water waveguide. For example, lower order modes tend to be trapped in the water column and are strongly affected by the details of the water sound speed profile, while higher order modes tend to penetrate more significantly into the bottom and are more strongly affected by the bottom properties. However, intermediate modes are less affected by the water sound speed profile and bottom properties. The mode-based approaches to the mismatch problem build on these considerations by rejecting lower order modes and/or higher order modes [10–12, 14, 15].

To proceed with the method developed here, the acoustic field is separated into groups of modes that experience the same kind of propagation (e.g. the group of modes that are refracted in the water column and have approximately the same turning point, the group of modes reflected at the surface and bottom that penetrate a similar distance into the bottom). The perturbations to different groups of modes caused by perturbations to environmental parameters may be considered uncorrelated because different mode groups are sensitive to the environment in different ways. Conversely, the perturbations to modes within a single group may be considered correlated and roughly predictable because they have similar sensitivity to environmental parameters. Further, for a specific group of modes a perturbation to environmental parameters may be considered roughly equivalent to
a perturbation in frequency. For instance, in generalized waveguide invariant theory changes in environmental parameters are associated to frequency shifts in acoustic intensity [38–43]. Such frequency shifts caused by dynamical processes inside the ocean (tides, internal waves, slope-water intrusions) have been observed in several experiments [39–43]. Although the above assumptions may not be precisely correct, they capture basic characteristics of shallow-water modal propagation sufficiently for our purposes, i.e., the localization algorithm based on them (developed below) is shown to be efficient and robust to environmental mismatch in simulations in Sec. IV.

2 Green’s function model

In this paper, the environmental model representing the best available knowledge is referred to as the assumed model, and the corresponding Green’s function as the assumed Green’s function, \( g_a \). The assumed model and assumed Green’s function are potentially erroneous, but we assume the truth lies somewhere around these initial assumptions. Therefore, the Green’s function is not considered to be a deterministic vector \( g_a \) but a random vector \( g \) in the \( N \)-dimensional space defined by the set of Green’s functions corresponding to all possible models. We assume that the probability distribution for \( g \) around \( g_a \) in the Green’s function space is defined by a covariance matrix \( \Sigma_g \) which has the form

\[
\Sigma_g = U \begin{pmatrix}
\sigma^2_{g_a} & 0 & \cdots & 0 \\
0 & \ddots & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & C_g
\end{pmatrix} U^H,
\]

(20)

with \( U = [\bar{g}_a, Q] \), where \( \bar{g}_a \) is the assumed Green’s function after normalization and \( Q \) is any \( N \times (N-1) \) matrix that forms an orthonormal basis in the space orthogonal to \( \bar{g}_a \). The matrix \( U^H \) is an orthonormal matrix which represents a rotation (change of basis) matrix from the standard basis to a basis defined by the vectors of \( U \) (e.g., \( z = U^H x \) is a vector whose coordinates are expressed in the basis defined by \( (u_1, u_2, \ldots, u_N) \) where \( U = [u_1, u_2, \ldots, u_N] \)). Thus, \( \sigma^2_{g_a} \) represents the...
variance in the direction of the assumed Green’s function. The ratio \( \sigma^2_{g_a} / \text{tr}(\Sigma_g) \) determines how much the true Green’s function is expected to depart from the assumed one; in other words, it characterizes the strength of its variability. The matrix \( C_g \) describes the covariance of the true Green’s function in the space orthogonal to the assumed Green’s function. It determines in which directions the true Green’s function may lie preferentially when it departs from the assumed one.

The considerations on the effect of mismatch introduced in Sec. III B 1 may be used to build the matrix \( C_g \) in Eq. (20) as follows (summarized in the flowchart of Fig. 2). This covariance matrix is approximated using the covariances of the Green’s function responses computed when the source frequency is varied for different groups of modes. The goal is to mimic the effect of mismatch on the Green’s function. First, groups of modes that experience the same kind of propagation are formed (see the previous section for an explanation or the following section for an example). The matrix \( C_g \) is divided into a sum of matrices \( C^k_g \), with each matrix corresponding to a group of modes \( k \). For each group of modes, the acoustic field is simulated for multiple frequencies in the neighborhood of the source frequency using the assumed model of the environment. These replicas are then projected on the space orthogonal to the assumed Green’s function \( g_a \) and used to build the empirical covariance matrix \( C^k_g \). Varying the frequency independently for each group of modes (the matrices \( C^k_g \) are summed) randomizes the phase which effectively decorrelates these groups. This procedure is justified in part since within a group of modes environmental perturbations may be roughly equivalent to frequency perturbations (see previous section). The range of frequency perturbations determines how the modes vary within a group. Different strategies may be adopted to set this range. The range of frequency perturbations may be determined from long term observations of frequency shifts caused by dynamical processes inside the ocean (tides, internal waves, water-slope intrusions, fronts...) in a given environment [39–43] and/or from the
generalized waveguide invariant values of some uncertain environmental parameters together with their expected variations [38, 40, 42]. Note that an alternative methodology here could be to carry out perturbations in specific environmental parameters, but frequency-based perturbations are more general. Similarly, determining an appropriate value for $\sigma_{ga}^2$ may be based on long term observations of the acoustic field in a particular situation or on some a priori knowledge of the degree of variability of the environment. In the latter case, $\sigma_{ga}^2$ may be obtained from a quick sensitivity analysis of the acoustic field conducted by varying randomly the environmental parameters in their expected range [44], with a sensitivity measure on the ratio of energy of the Green’s function staying in the assumed Green’s function direction $\bar{g}_a$ (i.e. on the ratio $\sigma_{ga}^2 / \text{tr}(\Sigma_g)$). The potential of the proposed approach to characterize the mismatch is validated by the results presented in the next section.

![Flow chart](image)

**Figure 2:** Flow chart of the construction of the matrix $C_g$. For each group of modes one simulates the acoustic field on the array at a few frequencies in the neighborhood of the source frequency, and projects the result on the space orthogonal to the assumed Green’s function. The computation of the empirical covariance in the basis defined by $Q$ gives a matrix $C^k_g$. The matrix $C_g$ is the sum of the matrices $C^k_g$. 

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IV  Simulation study

In this section the ability of the proposed Bayesian method to give reliable PPDs for source location is illustrated through a set of simulated examples, and the robustness of the method to environmental mismatch is assessed and compared to a mode-based approach through Monte-Carlo performance analyses.

A  Scenarios

The localization scenarios considered here are illustrated in Fig. 3. Two kinds of true environment are considered, both of which have an uncertain water column sound speed profile defined by the sound speeds $c_1$ (at the surface), $c_2$ (at 10 m depth), $c_3$ (at 50 m depth) and $c_4$ (at the seabed, depth $D$). The seabed for the first environment consists of an uncertain bottom halfspace while the second environment seabed consists of an uncertain sediment layer over an uncertain bottom halfspace. The parameter bounds for these two environments are given in Table 1. In all cases the assumed waveguide environment used to compute acoustic fields for trial source locations in the inversion is the first environment with all the environmental parameters taken to their central values (see Table 1).

![Figure 3: (Color online) Schematic diagram of the ocean environments for the source localization examples.](image)

The MFP inverse problem involves the localization of an acoustic source in range and depth.
The range is unknown with a uniform prior distribution on the interval $[500, 10000]$ m and the depth is unknown with a uniform prior distribution on the interval $[2, 95]$ m (the depth of the assumed environment is $D = 100$ m). Acoustic data are collected at an $N = 15$ elements vertical line array sampling the water column on a regular grid between $z = 2$ m and $z = 86$ m. The computation of the PPD is carried out for the likelihood with unknown variance in Eq. (8) on a numerical grid involving depth and range increments of 2 m and 50 m, respectively. The number of snapshots considered is $L = 10$. The SNR is the same for every snapshots and defined in decibels as $\text{SNR} = 10 \log \left( \frac{|n(f_m)|^2 \|g(f_m, \theta_0)\|^2}{N \sigma^2(f_m)} \right)$. Only the $f = 300$ Hz frequency is used to carry out the inversion. At this frequency, the acoustic field in the assumed waveguide consists of 16 propagating modes. Modal depth functions of the assumed waveguide are plotted in Fig. 4.

Figure 4: (Color online) Modal depth functions of the assumed waveguide environment.

Groups of modes are formed in order to design the matrix $C_g$. Considering that the groups of modes must be affected by different features of the waveguide, we form six groups of modes. Group 1 consists only of mode $m = 1$, group 2 consists only of mode $m = 2$, group 3 consists of modes $m = \{3 \cdots 7\}$, group 4 consists of modes $m = \{8 \cdots 12\}$, group 5 consists of modes $m = \{13, 14\}$ and group 6 consists of the last modes $m = \{15, M = 16\}$. Modes $m = \{1, 2\}$ are both trapped in the water column, but they are considered as uncorrelated because mode 2 is refracted much higher in the water column than mode 1 (see Fig. 4). Modes $m = \{13 \cdots M\}$
Le Gall

penetrate significantly in the bottom, but higher-order modes penetrate much more deeply in the bottom (see Fig. 4). It should be noted that the results are not highly sensitive to small changes in the divisions of the mode groups, so that this division can be somewhat arbitrary. Only the separation of mode \( m = 1 \) and mode \( m = 2 \) into distinct groups was found to be important.

Neighborhood frequencies in the range \( \pm 5\% \times f \) are considered for the construction of the matrix \( C_g \). It should be noted that frequency perturbation ranges between \( \pm 10\% \times f \) or \( \pm 1\% \times f \) also worked well. These values roughly correspond to the frequency shifts predicted by the depth-frequency waveguide invariant theory for the considered water depth uncertainty (the depth-frequency waveguide invariant is \( \gamma = -2 \), hence depth variations of \( \pm 2\% \) roughly correspond to frequency variations of \( \pm 4\% [38] \)). The variance \( \sigma^2_{ga} \) is determined from a sensitivity analysis of the acoustic field as described in Sec. III C with 50 random realizations of the first environment. The variance \( \sigma^2_{ga} \) is set so that the ratio \( \frac{\sigma^2_{ga}}{\text{tr}(\Sigma_g)} = 0.25 \), i.e. so that 25\% of the energy of the matrix \( \Sigma_g \) is in the direction of the assumed Green’s function. This value approximately characterizes the variability of the Green’s function when the environment varies, and need not be known precisely. Ratios between 0.15 and 0.75 were found to give close PPDs after marginalization most of the time (same location of the maxima and similar uncertainty).

B Test cases

The ability of the proposed method to give reliable PPDs for source localization is illustrated through three selected test cases which depict typical results of the localization process (we have carried out many more simulations). In each case the true environment used to generate the observed data is different from the assumed environment used in the localization. The true environmental parameters were chosen randomly and are given in Table 1. The SNR is 3 dB. The marginal PPD over source range and depth, integrated over the uncertain Green’s function, is computed with
the method proposed in this paper (mismatch integration) and compared to the PPD obtained with Bayesian MFP where the mismatch in the assumed environment is not accounted for (no mismatch integration is applied). The results are given in Fig. 5. In the first test case both methods find the correct source location with similar unimodal PPDs. In the second test case only the mismatch integration find the correct source location. Conventional MFP fails because of environmental mismatch whereas the proposed method is more robust. In the third test case neither method is able to precisely find the correct source location, although the mismatch integration method is close in both range and depth.

Note that in each plot the gray scale indicating probability of the joint marginal distribution is scaled according to the maximum probability. Only the mismatch integration method gives a reliable PPD in each case because it does not suggest a high certainty (i.e. a probability close to one) at an incorrect source location, as the classical technique does in test cases 2 and 3. Also, it does not simply gives uncertain PPDs. In the first test case the right position is suggested with a probability of 0.89. In the second test case the correct position is also identified with a good but slightly lower probability (0.48) which is not surprising because, as the conventional method does not work, the localization problem appears more challenging. In the third test case a wrong position is suggested but with a very low probability (0.039), so it is clear that the source localization result is highly uncertain.
Figure 5: (Color online) Joint marginal PPD over source range and depth together with the associated one-dimensional marginals for three test cases: (a) no integration over the uncertain Green’s function model, (b) integration over the uncertain Green’s function model. The circle indicates the MAP estimate, the triangle and the dashed lines indicate the true position.
C Monte-Carlo performance analyses

In this section, Monte-Carlo performance analyses are performed to assess the robustness of
the proposed method. The performance is evaluated in terms of probability of correct localization
(PCL) of the MAP estimate and root mean-squared error (RMSE) of the posterior mean estimate.
The PCL is defined as the probability that the absolute source localization error is 500 m or less
in range and 5 m or less in depth. For each SNR, \( N_{mc} = 5000 \) random realizations of the oceanic
waveguide, of source position, and of noise are generated. The PCL is a binomial-distributed
random variable with a standard deviation estimate of

\[
\sigma = \sqrt{\frac{\text{PCL} (1 - \text{PCL})}{N_{mc}}}. \tag{21}
\]

For \( N_{mc} = 5000 \) this standard deviation has a maximum value of 0.0071 at PCL = 0.5 which is
approximately the thickness of the plotted lines in Fig. 6, which shows the results. Two cases are
considered: in the first case the true oceanic waveguide is generated randomly as in environment
1 and in the second case the true oceanic waveguide is generated randomly as in environment 2
(see Fig. 3 and Table 1). The assumed waveguide used for the inversion is always environment 1
with all the environmental parameters taken to their central values; therefore using environment 2
to generate the data investigates if the localization process is robust to an error in the structure of
the assumed waveguide. The performance of the uncertain Green’s function approach is compared
to Bayesian MFP where the mismatch in the assumed environment is not accounted for, and to Liu
et al.’s [15] mode-based approach where unpredictable modes are consigned as noise (predictable
modes are taken to be modes \( m = \{3 \cdots 12\} \), which gave the best results amongst all possibilities).

Figure 6 shows that the mismatch integration outperforms the other methods and provides
a significant improvement in the PCL and the RMSE. It should be noted that the mode-based
approach [15] provides a significant improvement in PCL and in RMSE for source range over
regular MFP, but the RMSE for source depth is poorer for the mode-based approach than for regular MFP. This shows that this method may be less robust for depth discrimination because of the information that is lost by rejecting some lower order modes. Conversely, our method is robust for both the PCL and the RMSE.
Figure 6: (Color online) Localization performance study over 5000 Monte-Carlo simulations for the two test cases considered (left: environment 1, right: environment 2). The upper panel presents the results in term of PCL for the MAP estimate of range and depth. The two other panels present the results in term of RMSE for the posterior mean estimate of range and depth. The results obtained with the method presented in this article are depicted by continuous lines with crosses and compared to Liu et al. [15] mode-based approach (dashed lines) and regular MFP localization (continuous lines).
V Conclusion

In this paper, a novel Bayesian approach for MFP source localization has been proposed. The method handles uncertainty about the oceanic environment by considering the Green’s function of the waveguide as an uncertain random variable whose probability density accounts for environmental uncertainty. The integration of the PPD over the Green’s function probability density gives the marginal PPD over the localization parameters. Because brute-force integration is prohibitively costly in general, a technique has been developed to make the integration highly efficient. Some assumptions on modal propagation have been introduced to build a Green’s function probability density that considers the effect of environmental uncertainty on the Green’s function. In particular, decorrelation between mismatch for different groups of modes and similarity between mismatch and frequency shifts of the acoustic field have been taken into consideration. Monte-Carlo simulation results showed that the method is robust to mismatch in the environment. The method provided a significant improvement for both the PCL and the RMSE compared to conventional Bayesian MFP (where the mismatch in the assumed environment is not accounted for) and to a mode-based approach to the mismatch problem. Illustrations also showed that the method provides reliable PPDs for source location. When the method fails to localize the source it generally associates a low probability to the proposed position so that an analysis of the PPD may prevent taking wrong decisions.

Acknowledgments

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References


Table 1: Environmental parameters bounds for the two types of considered environments, assumed environment and true environments for the three test cases.
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1 (Color online) Schematic illustration of the Green’s function model in $N = 3$ dimensions and a real vector space. The axes $e_1, e_2, e_3$ represent the standard basis. The axes $u_1, u_2, u_3$ represent the eigenvectors of $\Sigma_g$ and the crosses represent normalized Green’s function generated from $\Sigma_g$. When the environment is uncertain, the Green’s function is no longer a deterministic vector but a random vector that may lie anywhere in the $N$ dimensional space. However, it generally retains some structure and lies preferentially in some regions of this space. In this schematic illustration, the Green’s functions are mostly oriented toward $u_1$ but may also direct toward $u_2$ while they direct very little toward $u_3$. This behavior is defined by the covariance matrix $\Sigma_g = U\Lambda U^H$, where $U = [u_1, u_2, u_3]$, $\Lambda = \text{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2)$ and $\sigma_1 > \sigma_2 \gg \sigma_3$.

2 Flow chart of the construction of the matrix $C_g$. For each group of modes one simulates the acoustic field on the array at a few frequencies in the neighborhood of the source frequency, and projects the result on the space orthogonal to the assumed Green’s function. The computation of the empirical covariance in the basis defined by $Q$ gives a matrix $C_{g}^{k}$. The matrix $C_g$ is the sum of the matrices $C_{g}^{k}$.

3 (Color online) Schematic diagram of the ocean environments for the source localization examples.

4 (Color online) Modal depth functions of the assumed waveguide environment.
Joint marginal PPD over source range and depth together with the associated one-dimensional marginals for three test cases: (a) no integration over the uncertain Green’s function model, (b) integration over the uncertain Green’s function model. The circle indicates the MAP estimate, the triangle and the dashed lines indicate the true position.

Localization performance study over 5000 Monte-Carlo simulations for the two test cases considered (left: environment 1, right: environment 2). The upper panel presents the results in term of PCL for the MAP estimate of range and depth. The two other panels present the results in term of RMSE for the posterior mean estimate of range and depth. The results obtained with the method presented in this article are depicted by continuous lines with crosses and compared to Liu et al. [15] mode-based approach (dashed lines) and regular MFP localization (continuous lines).