On Symmetric Alpha-Stable Noise after Short-Time Fourier Transformation

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Abstract—Statistical properties of real-valued symmetric α-stable noise after short-time Fourier transformation are derived. Circularity, stationarity and dependence between the real and imaginary components are studied as a function of the STFT parameters and the stability index α.

Index Terms—Alpha-stable, heavy-tailed distributions, short-time Fourier transform.

I. INTRODUCTION

S YMMETRIC alpha-stable distributions (SαS) are commonly employed to model impulsive perturbations of various physical origins, including underwater acoustic noise of snapping shrimp [1], man-made audio noise, as well as different types of electromagnetic phenomena [2]. Theoretical rationals for SαS models often result from its leptokurtic nature and are also provided by the generalized central limit theorem, which extends the central limit theorem to variables of infinite variance. For numerous signal processing applications, the discrete short-time Fourier transform (STFT) is widely used to develop tools dedicated to time-frequency analysis, such as the spectrogram, as well as those designed for heavy-tailed background noise [3]. In a statistical signal processing framework, knowledge of the distribution of the STFT coefficients is often required so as to optimize detection and estimation methods [4]. For instance, the STFT is a common sparse transform for locally harmonic signals and can be used to improve robustness of noise scale parameter estimation when observations do not result from noise only, but from the random presence of signals in independent and additive noise [5]. In this paper, we derive the characteristic function of independent and identically distributed (i.i.d.) real-valued SαS noise after STFT and study the properties of circularity, stationarity and dependence between the real and imaginary components as a function of the STFT and the SαS distribution parameters. This paper is organized as follows. Section II is devoted to the introduction of SαS distributions. Statistical properties of the STFT of SαS noise are derived in Section III, followed by conclusion in Section IV.

Notation: Throughout this paper, uppercase letters denote random variables, e.g. X, and uppercase boldface letters denote random vectors, e.g., X. The superscripts T stands for transposition. N(m,v) designates the distribution of a Gaussian random variable with mean m and variance v, and S(α,σ) the distribution of a SαS distribution with index of stability α and scale parameter σ. The least common multiple of two integers a and b is denoted by lcm(a, b) and the greatest common divisor by gcd(a, b). The uniform norm of a real-valued function f(x) is denoted ∥f(x)∥∞ = supx∈R |f(x)|. The notation α ≡ b (mod n) is used when the number a is equivalent (congruent) to the number b modulo n. Finally, the symbol Δ denotes equality in distribution and sgn the sign function.

II. SYMMETRIC ALPHA-STABLE DISTRIBUTIONS

The theory of SαS distributions is detailed in numerous references, see [6], [7]. Only definitions and properties relevant for our analysis are recalled in this section.

Definition 1 ([6]) A random variable X is stable if for any positive constants a and b, and for some positive c and some d ∈ R

\[ aX_1 + bX_2 \overset{d}{=} cX + d, \]

where X1 and X2 are independent copies of X. A random variable is symmetric stable if it is stable and symmetrically distributed around 0, e.g. X Δ= −X.

Property 1 ([6]) The characteristic function of a symmetric stable random variable X can be expressed as

\[ \Phi_X(u) = \exp(-|\sigma u|^{\alpha}), -\infty < u < +\infty \]

with 0 < α ≤ 2, and σ > 0.

The distribution of X is denoted by S(α,σ), where the index of stability α characterizes the decay rate of the distribution tails and the scale parameter σ measures dispersion or spread. One consequence of SαS heavy tails is that not all moments exist. More precisely, for 0 < α < 2, E{X|P} is finite only for 0 < p < α. With the exception of the Gaussian case (α = 2), SαS random variables have infinite variance. Also, note that closed-form expression of SαS probability density function are only available for Cauchy (α = 1) and Gaussian (α = 2) random variables.

Property 2 ([7]) Any SαS random variable X can be decomposed as X = A^{1/\alpha}G, where G and A are mutually
industrial, with $G \sim N(0, 2\sigma^2)$ and $\Lambda$ a right-skewed stable random variable, whose characteristic function is given in [7]. $X$ is also classified as sub-Gaussian and is said to be subordinated to $G$.

III. STATISTICAL PROPERTIES OF THE STFT OF SYMMETRIC ALPHA-STABLE NOISE

A. STFT

Let $\{X(m)\}_{m \in \mathbb{Z}}$ be a Stochastic random process with i.i.d. elements satisfying for all $m \in \mathbb{Z}$, $X(m) \sim S(\alpha, \sigma)$. Given a discrete analysis window $w(m)$ of length $M$, the STFT of $X(m)$ is defined as

$$X_{n,k} \triangleq \begin{bmatrix} X_{n,k}^r, X_{n,k}^i \end{bmatrix}_T,$$

where $\mathfrak{D} > k \geq 0$ and

$$X_{n,k}^r \triangleq \sum_{m=-D}^{nD+M-1} X(m) w(m-nD) \cos(\omega_km),$$

$$X_{n,k}^i \triangleq -\sum_{m=-D}^{nD+M-1} X(m) w(m-nD) \sin(\omega_km),$$

where $K > 0$ is the length of the discrete Fourier transform (DFT), $D > 0$ is the hop size between successive DFTs, and $\omega_k \triangleq 2\pi k/K, 0 \leq k \leq K-1$. $K/M$ corresponds to the factor of zero-padding. The indexes $n$ and $k$ refer to the discrete time and frequency locations, respectively. For the sake of readability, the summation range over $m$ is sometimes omitted. When not specified, it is equal to $\{nD, \cdots, nD + M - 1\}$.

B. Joint characteristic function

**Proposition 1** The joint characteristic function of $X_{n,k}$ can be expressed as

$$\Phi_{X_{n,k}}(u) = \exp \left( -\left( \sqrt{u^T u} \sigma \right)^\alpha \times s_{n,k}(u) \right),$$

where $s_{n,k}(u) = \sum_m |w(m-nD) \sin(\omega_km) - \theta(u)|^\alpha$, $u \triangleq [u_0, u_1]^T$ and

$$\theta(u) = -\frac{1}{2} \text{sgn}(u_0^2 - u_1^2) \arccos \left( \frac{-2u_0u_1}{u^T u} \right) - \frac{\pi}{4}.$$

**Proof:** Let $\hat{X}_{m,k} \triangleq X(m)[\cos(\omega_km), -\sin(\omega_km)]^T$. Using the independence of the $(X(m))’s$, we have

$$\Phi_{X_{n,k}}(u) \triangleq \mathbb{E} \left\{ \exp \left( iu^T \hat{X}_{n,k} \right) \right\} = \prod_m \Phi_{\hat{X}_{m,k}}(w(m-nD)u).$$

From Prop. 2, $X(m)$ is sub-Gaussian and can be expressed as $X(m) = A^{1/2}(m)\mathcal{G}(m)$. Therefore, $\hat{X}_{m,k}$ satisfies $\hat{X}_{m,k} = A^{1/2}(m)\mathcal{G}_{m,k}$, where $\mathcal{G}_{m,k}$ is a degenerate zero-mean Gaussian vector with covariance matrix

$$R_{m,k} = 2\sigma^2 \begin{bmatrix} \cos^2(\omega_km) & -\frac{1}{2} \sin(2\omega_km) \\ -\frac{1}{2} \sin(2\omega_km) & \sin^2(\omega_km) \end{bmatrix}.$$  

Using [7], [6, Eq. (8)], the characteristic function of $\hat{X}_{m,k}$ can then be expressed as

$$\Phi_{\hat{X}_{m,k}}(u) = \exp \left( -\frac{1}{2} u^T R_{m,k} u \right)^{\alpha/2}.$$  

From (6) and (8), we obtain

$$\Phi_{X_{n,k}}(u) = \exp \left( -\sum_m \left| \frac{1}{2} w^2(m-nD) u^T R_{m,k} u \right|^{\alpha/2} \right).$$

The expression of $\Phi_{X_{n,k}}(u)$ derives from

$$u^T R_{m,k} u \overset{(a)}{=} \sigma^2 (u_0^2 + u_1^2 + (u_0^2 - u_1^2) \cos(2\omega_km) - 2u_0u_1 \sin(2\omega_km)) \overset{(b)}{=} 2\sigma^2 u^T u \sin^2(\omega_km - \theta(u)),$$

where (a) is a straightforward expansion of $u^T R_{m,k} u$ and (b) is based on the following trigonometric identities: $\forall a, b, x \in \mathbb{R}, a \sin x + b \cos x = \sqrt{a^2 + b^2} \sin(x + \phi)$, where $\phi = \text{sgn}(b) \arccos \left( \frac{a}{\sqrt{a^2 + b^2}} \right)$ and $1 + \sin(2x) = 2 \sin^2(x + \frac{\pi}{4})$.

C. Stationarity and circularity

Due to the i.i.d. nature of the $X(m)$’s and the periodicities of trigonometric functions, the STFT output is expected to satisfy specific properties related to stationarity and circularity. Hereafter, we detail such properties in the SoS case.

**Corollary 1** $X_{n,k}$ is cyclo-stationary in $n$ such that

$$\Phi_{X_{n,k}}(u) = \Phi_{X_{n+p,k}}(u),$$

where

$$P = \begin{cases} \text{lcm}(2KD, K) \quad &\text{if } k > 0 \\ 1 \quad &\text{if } k = 0. \end{cases}$$

**Proof:** By a change of variable, $s_{n,k}(u)$ can be written as $s_{n,k}(u) = \sum_{m=0}^{M-1} |w(m)|^\alpha |\sin(\omega_km + nD) - \theta(u)|^\alpha$. Moreover, since $|\sin(k\pi)|$ is $\pi$-periodic and by noticing that $P$ is the smallest integer such that $\omega_mDP \equiv 0 \pmod{\pi}$, we conclude that $\Phi_{X_{n,k}}(u) = \Phi_{X_{n+p,k}}(u)$.

**Corollary 2** For all analysis window $w(\cdot)$ and all strictly positive integers $M$ and $\ell$, $X_{n,k}$ is stationary in $n$ when $K$ is even and when the hop size $D$ satisfies

$$D = \ell \frac{K}{2}.$$  

**Proof:** Corollary 2 is obtained by simply noticing that for $D = \ell \frac{K}{2}$, we have $P = 1$.

Let $X_{n,k}^z$ be defined as

$$X_{n,k}^z \triangleq \begin{bmatrix} \cos z & -\sin z \\ \sin z & \cos z \end{bmatrix} X_{n,k}.$$  

By definition, $X_{n,k}$ is circular if $X_{n,k}^z \overset{d}{=} X_{n,k}$, for all $z \in \mathbb{R}$. As shown in the following corollary, the properties of circularity and stationarity are closely related.

**Corollary 3** A sufficient condition for strict stationarity in $n$ is circularity. In other words, if $X_{n,k}$ is circular then $X_{n,k} \overset{d}{=} X_{n+\ell,k}$, for all $\ell \in \mathbb{N}$. 

Proof: It can be shown that the characteristic function of $X_{n,k}^z$ is expressed as
$$
\Phi_{X_{n,k}^z}(u) = \exp \left( - \sqrt{u^T u} \sigma \right) \times s_{n,k}^z(u),
$$
(14)
where
$$
s_{n,k}^z(u) = \sum_{m=0}^{M-1} |w(m)| \sin(\omega_k (m + nD) + z - \theta(u))|^\alpha.
$$
(15)

Therefore, $\Phi_{X_{n,k}^z}(u) = \Phi_{X_{n,k}^z}(z)$, for $z = \omega_k D \ell$. Assuming that $X_{n,k}$ is circular, we have $\Phi_{X_{n,k}^z}(u) = \Phi_{X_{n,k}^z}(u)$, for all $z$ in $\mathbb{R}$, which concludes the proof. ■

Note that the converse of Corollary 3 is not true.

In numerous applications, detection/estimation algorithms rely on the assumption that the input data are both stationary and circular. According to (14) and (15), circularity is obtained if and only if $s_{n,k}^z(u) = s_{n,k}(u)$, for all $z$. In the general case, this equality does not hold, so that $X_{n,k}$ is not circular. However, by carefully choosing the STFT parameters, a rotational symmetry of the distribution of $X_{n,k}$ can be obtained, leading, in some cases, to an approximate circularity and therefore to an approximate stationarity.

1) The boxcar window: Let us consider the boxcar window defined as $w(m) = 1/\sqrt{M}$, $m \in \{0, \ldots, M-1\}$. In that case, $s_{n,k}(u) = \sum_{m=0}^{M-1} \sin(\omega_k (m + nD) - \theta(u))|^\alpha$. By noticing that $|\sin(\omega_k (m + nD) - \theta(u))|^\alpha$ is a periodic function of $m$, with a period $P_z = \text{lcm}(2k, K)/(2k)$, if $M$ is chosen to be a multiple of $P_z$, we obtain
$$
s_{n,k}(u) = s_{n,k}^z(u), \forall z \equiv 0 \pmod{\pi/P_z}.
$$
(16)

This leads us to the following corollary.

Corollary 4 For a boxcar analysis window of length $M = \ell \times P_z$, with $\ell \in \mathbb{N}^+$ and $P_z = \text{lcm}(2k, K)/(2k)$, $X_{n,k}$ presents a rotational symmetry of order $P_z$, so that
$$
X_{n,k}^d = X_{n,k}^z, \forall z \equiv 0 \pmod{\pi/P_z}.
$$
(17)

Note that this result holds for all $0 < \alpha \leq 2$, $n \in \mathbb{N}$ and $D \in \mathbb{N}^+$. Figure 1-(a) illustrates this rotational symmetry for $\alpha = 0.5$, $M = K = 8$, and for the frequency indexes $k = 1$, leading to $P_z = 4$.

In the specific Gaussian case ($\alpha = 2$), it can be shown that the STFT parameters described in Corollary 4 lead actually to a strictly circular vector $X_{n,k}$.

Circularity is strictly obtained when $P_z = +\infty$. However, in practice, circularity is a reasonable assumption when $P_z$ is large, leading to a near-isotropic distribution. By definition, $P_z$ is bounded by $1 \leq P_z \leq K$ and depends both on the frequency index $k$ and the length $K$ of the DFT. Choosing a large value $K$ does not guarantee that $P_z$ is large for all $k$. For instance, if we set $K = 2^N$ and $k = K/4$, we have $P_z = 2$, for all integer $N \geq 2$. In fact, $P_z$ is maximal and equal to $K$ when $2k$ and $K$ are coprime. This can be verified by expressing $P_z$ as
$$
P_z = \frac{\text{lcm}(2k, K)}{2k} = \frac{K}{\gcd(2k, K)},
$$
(18)
with $\gcd(2k, K) = 1$, if $2k$ and $K$ are coprime.

As consequence of (18), it can be shown that for a boxcar window, with a zero-padding factor set to 1, $P_z = K, \forall 0 < k \leq K - 1$ iff the DFT length $K$ is chosen to be a prime number. Choosing this set of parameters, with $K$ prime, also implies that the $X_{n,k}$ are identically distributed across frequencies. In addition, if $K$ is chosen to be sufficiently large, the characteristic function becomes near-isotropic and the approximation of circularity and therefore of stationarity then becomes reasonable. This is illustrated in Figure 1-(b), where the characteristic function of $X_{n,k}$ is plotted for $\alpha = 0.5$ with $M = K = 127$.

2) The other windows: So far, we have shown that a rotational symmetry can be obtained with a boxcar window, thanks to the periodicity described by (16). This periodicity is not strictly satisfied for common windows such as Hanning, Gauss, Blackman, etc. However, as shown in Table I, for most windows, $s_{n,k}^z(u)$ is almost-periodic and satisfies
$$
\|s_{n,k}^z(u) - s_{n,k}^{z+\pi/P_z}(u)\|_\infty \leq \epsilon,
$$
(19)

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The last implication results from the fact when $K = M = P_z$, with $K$ prime, $s_{n,k}(u) = s_{n,k'}(u)$, for all $k$ and $k'$ in $\{1, \ldots, K - 1\}$. 

![Figure 1](image.png)

Fig. 1. Illustration of the rotational symmetry and near-circularity of $X_{n,k}$, $(\alpha = 0.5, \sigma = 1, k = 1)$. (a) and (b), boxcar window, with $K = 8$ and $K = 127$, respectively. (c) Hanning window, with $K = 8$. 

INDEX OF “ALMOST-PERIODICITY”: $\|s_{n,k}^z(u) - s_{n,k}^{z+\pi/P_z}(u)\|_\infty/\|s_{n,k}^z(u)\|_\infty$, FOR $\alpha = 0.5, k = 1, n = 0$. 

1The specific Gaussian case ($\alpha = 2$), it can be shown that the STFT parameters described in Corollary 4 lead actually to a strictly circular vector $X_{n,k}$. 

2The last implication results from the fact when $K = M = P_z$, with $K$ prime, $s_{n,k}(u) = s_{n,k'}(u)$, for all $k$ and $k'$ in $\{1, \ldots, K - 1\}$. 

Table 1 

- **Box.** 
- **Hann.** 
- **Gauss.** 
- **Black.** 

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where $\epsilon < \| s_{\alpha,k}^2(u) \|_{\infty}$. This “almost-periodicity” implies that $\Phi_{X_{n,k}}(u)$ presents an “almost” rotational symmetry of order $P_z$. This “almost” symmetry is visible when the characteristic function is plotted as in Figure 1-(c).

D. Marginal characteristic functions and dependence

**Corollary 5** $X_{n,k}^r$ and $X_{n,k}^i$ are SoS random variables, whose distributions are $X_{n,k}^r \sim S(\alpha, \sigma_r)$ and $X_{n,k}^i \sim S(\alpha, \sigma_i)$, respectively, with

\[
\sigma_r = \sigma \left( \sum_{m} |w(m-nD)\cos(\omega_km)|^\alpha \right)^{1/\alpha}
\]

\[
\sigma_i = \sigma \left( \sum_{m} |w(m-nD)\sin(\omega_km)|^\alpha \right)^{1/\alpha}
\]

$X_{n,k}^r$ and $X_{n,k}^i$ are identically distributed when their respective scale factors are equal. Based on our previous discussion in Section III-C, it can be shown that strict equality between $\sigma_r$ and $\sigma_i$ is obtained for all $0 < \alpha \leq 2$, when $X_{n,k}$ presents a rotational symmetry of order $P_z$, with $P_z$ even.\(^3\) Also, note that the “almost” symmetry of Section III-C, with $P_z$ even, leads to an “almost” equality between $\sigma_r$ and $\sigma_i$. This is also the case when $K$ is large and prime. For numerous sets of STFT parameters, the difference between $\sigma_r$ and $\sigma_i$ can be assumed to be negligible for practical applications.

Although $X_{n,k}^r$ and $X_{n,k}^i$ can be identically distributed, they are not independent in the general case. A case of independence also occurs when $X_{n,k}^r$ and $X_{n,k}^i$ are assumed to be negligible for practical applications.

The level of dependence between $X_{n,k}^r$ and $X_{n,k}^i$ can be quantitatively assessed by computing the $p$-distance measure\(^9\). Suppose that $E \{ |X_{n,k}^r|^p \} + E \{ |X_{n,k}^i|^p \} < +\infty$. Let $V^p$ denote the $p$-distance covariance defined by

\[
V^p \left( \Phi_{X_{n,k}^r}, \Phi_{X_{n,k}^i} \right) \Delta \gamma \int_{R^2} \left| \frac{\Phi_{X_{n,k}}(u) - \Phi_{X_{n,k}}(u_0)\Phi_{X_{n,k}}(u_1)}{|u_0|^{1+p}|u_1|^{1+p}} \right|^2 du,
\]

where $\gamma$ is some suitable constant (see\(^9\), Lemma 1). The $p$-distance measure is expressed as

\[
D^p = \left( \frac{V^p \left( \Phi_{X_{n,k}^r}, \Phi_{X_{n,k}^i} \right)}{\sqrt{V^p \left( \Phi_{X_{n,k}^r}, \Phi_{X_{n,k}^r} \right) V^p \left( \Phi_{X_{n,k}^i}, \Phi_{X_{n,k}^i} \right)}} \right)^{1/2},
\]

if $V^p \left( \Phi_{X_{n,k}^r}, \Phi_{X_{n,k}^i} \right) V^p \left( \Phi_{X_{n,k}^i}, \Phi_{X_{n,k}^r} \right) > 0$ and $D^p = 0$ otherwise. According to\(^9\), this measure equals zero only if the real and the imaginary components are independent.

\(^3\) Note that in the specific Gaussian case ($\alpha = 2$), $P_z$ is not required to be even.

Figure 2 shows the $p$-distance as a function of the stability index $\alpha$ for different analysis windows and DFT lengths. While we have shown that circularity, stationarity, and identical distributions between $X_{n,k}^r$ and $X_{n,k}^i$ can be assumed in many scenarios, numerical results indicate that it is not the case for independence, especially for small values of $\alpha$.

Figure 2 also highlights the impact of the analysis window, which becomes significant for small values of $K$ as $\alpha$ gets closer to 2 and almost negligible for large value of $K$. This can be explained by the fact that the property of dependence becomes strongly interlinked with the one of circularity as $\alpha$ tends to 2. We recall that in the Gaussian case, circularity implies independence between $X_{n,k}^r$ and $X_{n,k}^i$.

IV. CONCLUSIONS

In the general case, we have shown that STFT coefficients of i.i.d real-valued SoS noise are cyclo-stationary, non-circular, and have non i.i.d real and imaginary components. However, with specific STFT parameters, based on the boxcar window for instance and/or on large DFTs of prime sizes, it is possible to obtain “almost” circularity and stationarity. Dependence between the real and the imaginary components can only be achieved in the Gaussian case or for a normalized frequency of 1/4.

REFERENCES


