Matched-Field Performance Prediction with Model Mismatch

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Abstract—Matched-field estimation is known to be sensitive to mismatch between the assumed replica of the acoustic field and the actual field. An interval error-based method (MIE) is proposed to predict the mean-squared error (MSE) performance for multisnapshot and multifrequency maximum-likelihood matched-field estimation under model mismatch. The source signal is assumed deterministic unknown. Global errors are predicted by deriving exact expressions of pairwise error probabilities with model mismatch in conjunction with the use of the Union bound. Local errors are approximated using a Taylor expansion of the MSE. Numerical examples show the accuracy of the method.

Index Terms—Matched-field processing, mismatch, method of interval errors, underwater acoustics

I. INTRODUCTION

In underwater acoustics, nonlinear inverse problems are commonly solved using Matched-field processing (MFP) [1]. From data collected with an array of sensors, MFP compares measured acoustic fields with replica fields computed by a propagation model. Estimates of parameters such as source position, geoacoustic and/or oceanographic quantities are obtained by maximizing a cost function that quantifies the match between the measured data and the simulated replicas.

Noise in the measured data as well as mismatch between the assumed replica fields and the actual field may result in poor estimation performance. MFP estimation suffers from large ambiguities that lead to non-local errors at low signal-to-noise ratio (SNR) and mismatch has a serious effect on performance, particularly at high SNR. Therefore, it is important to quantify the performance of MFP to know and understand its limitation as well as to determine the operational conditions required to obtain acceptable estimates.

The performance prediction of MFP is not a simple problem if we wish to avoid computationally intensive Monte-Carlo simulations. MFP performance analysis is commonly carried out through the computation of limiting bounds on the mean-square error (MSE) [2]–[5] or by using the method of interval errors analysis (MIE) [5]–[8]. MIE does not bound the performance but approximates the MSE by capturing local as well as global errors for a particular estimator [9]–[12]. To the best of our knowledge, performance analysis in the presence of model mismatch has never been considered for the full multisnapshot and multifrequency MFP problem under the deterministic signal model (conditional model) where the source signals are assumed deterministic unknown; although this data model is very common for parameter estimation in the maximum-likelihood MFP framework [13], [14]. The aforementioned references taking mismatch into account consider either a monofrequency stochastic model (unconditional model) [7] or a particular case of the monofrequency deterministic model where relative source spectral information is available between snapshots [10]. In this paper, the MIE approach is derived to predict MFP MSE performance in the presence of model mismatch for the full multisnapshot and multifrequency MFP problem under the deterministic signal model. Note that the approach in [7], [12] accounts for an unknown noise covariance matrix whereas this latter is assumed known here.

The paper is organized as follows. The observation model and the MFP estimator are described in Section II. Section III presents the MIE analysis corresponding to our MFP context. Numerical examples illustrating the results are shown in Section IV, followed by conclusions in Section V.

Notation: Throughout this paper, lowercase boldface letters denote vectors, e.g., $x$, and uppercase boldface letters denote matrices, e.g., $A$. The superscripts $^T$ and $^H$ mean transposition and Hermitian transposition, respectively. $I_N$ and $0_{N,M}$ denote the $N \times N$ identity matrix and the $N \times M$ zero matrix, respectively. $\text{Re} [\cdot]$ denotes real part of a complex quantity. Operator $\| \cdot \|$ designates the standard Euclidean norm. Finally, $\mathbb{E} \{ \cdot \}$ denotes expectation and $\mathbb{P}$ denotes the probability measure.

II. OBSERVATION MODEL AND MAXIMUM LIKELIHOOD ESTIMATE

Consider an array of $N$ sensors in the ocean acoustic waveguide. The goal of MFP is to estimate a set of unknown parameters $\theta$ from the array measurements. This parameter set may be the source position (range, depth, bearing...) for source localization, and/or ocean environmental parameters (bathymetry, sound speed, density...) for tomography and geoacoustic inversion. For each snapshot $l$ and each frequency $m$, the complex array output is modeled by the $N \times 1$ vector

$$y_l(f_m) = s_l(f_m) \cdot g(f_m, \theta) + w_l(f_m),$$

$$l = 1, ..., L, \ m = 1, ..., M \ (1)$$

where,

- $L$ is the number of snapshots available for each of the $M$ frequencies.
• $g(f_m, \theta)$ is a complex $N \times 1$ vector representing the transfer function of the medium at frequency $f_m$ for the propagation from the source to each of the $N$ receivers. It is usually termed Green’s function.
• $s_l(f_m)$ is an unknown deterministic complex scalar representing the source amplitude and phase.
• $w_l(f_m)$ is a complex $N \times 1$ vector representing a circularly-symmetric, zero mean Gaussian noise that is independent of the source signal. It is assumed to be white in frequency and time with positive definite spatial covariance matrix $\Sigma_w(f_m) = \sigma_w^2(f_m)C_w(f_m)$. This matrix is assumed known and to simplify subsequent derivations and analysis we will work in whitened coordinates and assume that $\Sigma_w(f_m) = \sigma_w^2(f_m)I_N$.

The set of observations is collected in the following vector

$$y = [y_1^T(f_1) \ldots y_L^T(f_1) \ldots y_L^T(f_M) \ldots y_L^T(f_M)]^T.$$ (2)

To estimate $\theta$, matched-field methods compare the data measured on the array with replicas of the acoustic field derived from the wave equation. Model mismatch arises when these replicas differ from the actual field. This occurs for instance when the assumed oceanic waveguide differs from the true one or when system parameters are erroneous (inaccurate receivers’ positions...). Formally, the assumed Green’s functions $g_r(f_m, \theta)$ used to estimate $\theta$ differs from the true Green’s function $g(f_m, \theta)$. When $s_l(f_m)$ is assumed unknown and $\Sigma_w(f_m)$ is known, the ML estimate of the parameter set $\theta$ under model mismatch is [14]

$$\hat{\theta} = \arg \max_{\theta} C(\theta),$$ (3)

where

$$C(\theta) = \sum_{m=1}^{M} \sum_{l=1}^{L} \frac{|q_l^H(f_m, \theta)R_{l,m}q_l(f_m, \theta)|}{\sigma_w^2(f_m)},$$ (4)

$$R_{l,m} = y_l(f_m)y_l^H(f_m)$$ and $q_l(f_m, \theta) = \frac{q_l(f_m, \theta)}{||q_l(f_m, \theta)||}$.

If the true set of parameters is denoted as $\theta_0$, the search function $C(\theta)$ with a noise-free observation is called the ambiguity function (AF) $\psi(\theta)$ and satisfies

$$\psi(\theta) = \sum_{m=1}^{M} \sum_{l=1}^{L} \gamma(l, f_m)|g_r^H(f_m, \theta)g_l(f_m, \theta_0)|^2,$$ (5)

where $\gamma(l, f_m) = \frac{|g_l(f_m, \theta)|}{\sigma_w^2(f_m)}||g_l(f_m, \theta_0)||^2$ and $g_l(f_m, \theta_0) = \frac{g_l(f_m, \theta_0)}{||g_l(f_m, \theta_0)||}$.

The study of this AF helps to understand the impact of model mismatch on the estimation performance. An example is shown in Fig. 1 for a range estimation problem. It is observed that:

- the maximum of the AF with mismatch is at a value $\theta_{mis}$ different from the true value $\theta_0$; the estimator will be biased,
- the mainlobe is wider than in the no-mismatch case: the local error around $\theta_{mis}$ will be larger than the Cramér-Rao bound (CRB) [15],
- the amplitude ratio between the mainlobe and the sidelobes is smaller than in the no-mismatch case: the “threshold region” where global errors dominate will start at larger SNR.

![Fig. 1. Example of ambiguity functions with and without model mismatch](image)

Note that these observations may not hold true in all cases, they are scenario-dependent but typically occurred for the simulated cases considered by the authors that focused on shallow water environments (pekeris waveguides, summer water sound speed profiles, sediment layers over a basement).

### III. Method of Interval Errors

MIE provides an approximation of the mean square error (MSE) for a given estimator (see the tutorial treatment by Van Trees and Bell in [16]). As shown in Fig. 1, the AF usually exhibits a typical mainlobe/sidelobe behavior: MIE builds upon the decomposition of the MSE into two terms: the local errors that concentrate around the mainlobe peak and outliers that concentrate around the sidelobe peaks. Consider the true parameter $\theta_0$ and a discrete set of $N_o$ parameter points $\{\theta_1, \theta_2, \ldots, \theta_{N_o}\}$ sampled at the sidelobe maxima of the AF with mismatch. The conditional MSE of the ML estimator can then be approximated as [8]–[12]:

$$\mathbb{E}_y \left\{ (\hat{\theta} - \theta_0)(\hat{\theta} - \theta_0)^T \right\} \approx \mathbf{MSE}^{(\text{local})}(\theta_0) + \sum_{k=1}^{N_o} P_c(\theta_k|\theta_{\text{mis}}) \times \mathbf{MSE}^{(\text{local})}(\theta_k) \times (\hat{\theta} - \theta_0)(\hat{\theta} - \theta_0)^T,$$ (6)

where $P_c(\theta_k|\theta_{\text{mis}})$ is the pairwise error probability of the ML estimator under model mismatch (3), i.e. the probability of deciding in favor of the parameter $\theta_k$ in the binary hypothesis test $\{\theta_k, \theta_{\text{mis}}\}$. The pairwise error probability $P_c(\theta_k|\theta_{\text{mis}})$ is used as an approximation of the probability that the estimate falls on the sidelobe $k$ and $(1 - \sum_{k=1}^{N_o} P_c(\theta_k|\theta_{\text{mis}}))$ is used as an approximation of the probability that the estimate falls on the mainlobe (i.e. the probability of local errors). $\mathbf{MSE}^{(\text{local})}(\theta_0)$ is the asymptotic MSE of the ML estimator. In the absence of mismatch, the CRB is usually a good predictor of the performance in this region [5]. Since mismatch is possible, these local errors must be approximated by other means as described in III-B.
Note that the MIE approximation begins to over predict the MSE at low SNR. Thus, it is hard limited not to exceed the variance of a uniform distribution in the search interval of the ML estimator.

A. Pairwise error probability

The pairwise error probability $P_e(\theta_k|\theta_{\text{mis}})$ of the ML estimator under model mismatch (3) is given by

$$P_e(\theta_k|\theta_{\text{mis}}) = \mathbb{P} \left( \sum_{m=1}^{M} \sum_{l=1}^{L} \left| \frac{\mathbb{E}(f_m, \theta_{\text{mis}}) Y_l(f_m)}{\sigma_w^2(f_m)} \right|^2 - \sum_{m=1}^{M} \sum_{l=1}^{L} \left| \frac{\mathbb{E}(f_m, \theta_{\text{mis}}) Y_l(f_m)}{\sigma_w^2(f_m)} \right|^2 \geq 0 \right).$$ (7)

Following the same approach as in [5, Sec. V-C], this pairwise error probability can be expressed as

$$P_e(\theta_k|\theta_{\text{mis}}) = 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + \lambda_{1m}(j\omega + \beta)} e^{-c} d\omega, \quad (8)$$

$$c = \sum_{m=1}^{M} \sum_{l=1}^{L} \frac{\lambda_{1m}\mu_{1m,l}^2(j\omega + \beta)}{(1 + \lambda_{1m}(j\omega + \beta))} + \frac{\lambda_{2m}\mu_{2m,l}^2(j\omega + \beta)}{(1 + \lambda_{2m}(j\omega + \beta))}, \quad (9)$$

for some $\beta > 0$ such that $1 + \beta \lambda_{1m} > 0$ and $1 + \beta \lambda_{2m} > 0$. $\lambda_{1m,2m}$ are the non-zero eigenvalues of

$$\mathbb{E}(f_m, \theta_{\text{mis}}) \mathbb{E}(f_m, \theta_{\text{mis}}) - \mathbb{E}(f_m, \theta_{\text{mis}}) \mathbb{E}(f_m, \theta_{\text{mis}})$$ (10)

and satisfy

$$\lambda_{1m,2m} = \pm \sqrt{1 - \left| \mathbb{E}(f_m, \theta_{\text{mis}}) \mathbb{E}(f_m, \theta_{\text{mis}}) \right|^2}. \quad (11)$$

The values $\mu_{1m,l}$, $\mu_{2m,l}$ can be expressed as a function of the SNR $\gamma(l, f_m)$. In the general case with possible mismatch they satisfy

$$\mu_{1m,l} = \sqrt{\frac{\gamma(l, f_m)(1 - \lambda_{1m,2m})}{2 \lambda_{1m,2m}^2}} \times$$

$$\left( \mathbb{E}(f_m, \theta_{\text{mis}}) \mathbb{E}(f_m, \theta_{\text{mis}}) - \frac{1}{1 - \lambda_{1m,2m}^2} \times \mathbb{E}(f_m, \theta_{\text{mis}}) \mathbb{E}(f_m, \theta_{\text{mis}}) \right). \quad (12)$$

B. Asymptotic local MSE

When expanded, the MSE can be expressed as

$$\mathbb{E}(\hat{\theta} - \theta_0)(\hat{\theta} - \theta_0)^T = \mathbb{E}_Y \left\{ \Delta \hat{\theta} \Delta \hat{\theta}^T \right\} + \mathbb{E}_Y \left\{ \Delta \hat{\theta} \right\} b^T(\theta_0)$$

$$+ \ b(\theta_0) \mathbb{E}(\Delta \hat{\theta})^T + b(\theta_0)b^T(\theta_0),$$

where $\Delta \hat{\theta} = \hat{\theta} - \theta_{\text{mis}}$ and $b(\theta_0) = \theta_{\text{mis}} - \theta_0$. The value $\theta_{\text{mis}}$ as well as the bias $b(\theta_0)$ can easily be obtained by analyzing the AF with mismatch. The asymptotic local MSE is then derived by computing an asymptotic approximation of $\mathbb{E}_Y \left\{ \Delta \hat{\theta} \Delta \hat{\theta}^T \right\}$ and $\mathbb{E}_Y \left\{ \Delta \hat{\theta} \right\}$ using the same strategy as in [17] and [18], except that asymptotic refers to the SNR and not to the number of snapshots (this is more convenient for the deterministic signal model and is also justified by the fact that few snapshots are usually available in MFP applications).

More precisely, let $\theta_i$ denote the $i$-th entry of $\theta$, $\Delta \hat{\theta}$ is approximated using a Taylor expansion of $\frac{\partial \mathbb{E}}{\partial \theta_i}$ at infinite SNR, i.e., at $\theta_{\text{mis}}$ and $R_{\text{mis}} = |s_{l,m}|^2 g(f_m, \theta_0) g^H(f_m, \theta_0)$:

$$\frac{\partial \mathbb{E}}{\partial \theta_i} (\hat{\theta}_i, R_{\text{mis}}) \approx \frac{\partial \mathbb{E}}{\partial \theta_i} (\theta_{\text{mis}}, R_{\text{mis}}) + \frac{\partial^2 \mathbb{E}}{\partial \theta^2} (\theta_{\text{mis}}, R_{\text{mis}}) \Delta \hat{\theta}$$

$$+ \sum_{m=1}^{M} \sum_{l=1}^{L} \text{tr} \left\{ \frac{\partial^2 \mathbb{E}}{\partial \theta_i \partial \theta_i} (\theta_{\text{mis}}, R_{\text{mis}}) \right\} \Delta R_{l,m},$$

where $\Delta R_{l,m} = R_{l,m} - R_{\text{mis}}$. The ML estimate $\hat{\theta}$ is obtained with the observed data $R_{l,m}$, and the value $\theta_{\text{mis}}$ corresponds to the output of the ML estimator with infinite SNR therefore

$$\frac{\partial \mathbb{E}}{\partial \theta} (\hat{\theta}, R_{l,m}) = 0_{N \times 1}, \quad (15)$$

$$\frac{\partial \mathbb{E}}{\partial \theta} (\theta_{\text{mis}}, R_{\text{mis}}) = 0_{N \times 1}. \quad (16)$$

Since $\Delta R_{l,m}$ is Hermitian, it can be shown that

$$\text{tr} \left\{ \frac{\partial^2 \mathbb{E}}{\partial \theta \partial \theta} (\theta_{\text{mis}}, R_{\text{mis}}) \right\} \Delta R_{l,m} \right\} =$$

$$\text{Re} \left[ \frac{2}{\sigma_w^2(f_m)} \mathbb{E}(f_m, \theta_{\text{mis}}) \mathbb{E}(f_m, \theta_{\text{mis}}) \right]. \quad (17)$$

Therefore, from (14), for $i = 1 \cdots N$,

$$\Delta \hat{\theta} \approx \left( \frac{\partial \mathbb{E}}{\partial \theta} (\theta_{\text{mis}}, R_{\text{mis}}) \right)^{-1} \times$$

$$\text{Re} \left[ \frac{2}{\sigma_w^2(f_m)} \mathbb{E}(f_m, \theta_{\text{mis}}) \mathbb{E}(f_m, \theta_{\text{mis}}) \right] \Delta R_{l,m}. \quad (18)$$

After some tedious algebra, $\mathbb{E}_Y \left\{ \Delta \hat{\theta} \Delta \hat{\theta}^T \right\}$ and $\mathbb{E}_Y \left\{ \Delta \hat{\theta} \right\}$ can be obtained from (18). Define

$$a_m = \mathbb{E}(f_m, \theta_{\text{mis}}), \ c_m = g(f_m, \theta_0), \ D_m = \frac{\partial^2 \mathbb{E}}{\partial \theta_i \partial \theta_i} (f_m, \theta_{\text{mis}}),$$

and the $N \times N$ matrix

$$F_{i,j} = \sum_{m=1}^{M} \sum_{l=1}^{L} \frac{|s_{l,m}|^2}{\sigma_w^2(f_m)} \text{Re} \left[ \frac{\partial^2 a_m}{\partial \theta_j \partial \theta_i} \right] c_m c_m^H d_m$$

$$+ \frac{\partial a_m}{\partial \theta_i}^H c_m c_m^H \frac{\partial a_m}{\partial \theta_j}. \quad (19)$$

We then obtain

$$\mathbb{E}_Y \left\{ \Delta \hat{\theta} \right\} = -F^{-1} \text{Re} \left[ \sum_{m=1}^{M} \sum_{l=1}^{L} D_m a_m \right], \quad (20)$$

$^{1}$The estimates are local maxima (except in the particular case where they are on the edge of the parameter search space, this case is not considered).
The theoretical results are illustrated for MFP source range estimation in a shallow water environment. The MFP configuration is presented in Fig. 2. The waveguide is a Pekeris waveguide [19] which consists in an isospeed water layer overlaying a semi-infinite fluid basement. This waveguide is classically used to model shallow water environments. The ML estimator (3) assumes that the Pekeris waveguide has the parameters given by the assumed environment in table I. The source depth is assumed known and is fixed to $z_s = 30 \text{ m}$, whereas the range is unknown in the interval $r_s = [4, 6] \text{ km}$. The receiving array is a vertical array of $N = 12$ elements linearly spaced between $z_1 = 5 \text{ m}$ and $z_N = 95 \text{ m}$. We assume that we observe $L = 5$ snapshots of the source signal as well as $M = 8$ frequencies logarithmically spaced from 50 – 500 Hz. The Green’s functions are computed using normal mode theory [20].

Two mismatch scenarios are considered. In the first one, the source range is $\theta_0 = 5000 \text{ m}$ and the actual environment is given by scenario 1 in table I. In the second one, the source range is $\theta_0 = 5100 \text{ m}$ and the actual environment is given by scenario 2 in table I. Monte-Carlo simulations are performed with 5000 iterations for each SNR. Since the asymptotic MSE can be dominated by the bias $b(\theta_0)$, results are also presented for the bias-free MSE as expressed by $\mathbb{E}_x \left\{ \Delta \hat{\theta} \Delta \hat{\theta}^T \right\}$ in (22).

The results are shown in Fig. 3 and 4 for the first and second scenario, respectively. As a reference, the CRB is also plotted. In both cases, MIE accurately predicts the threshold region as well as the asymptotic performance of the ML estimator (3) with model mismatch. The second scenario is quite unusual: due to mismatch, the mainlobe gets split into two lobes with close amplitudes which explains the notable increase of the bias-free MSE for SNRs lower than 15 dB. In both scenarios, the impact of mismatch is significant: the bias limits the high SNR performance and, because mismatch widens the mainlobe, the bias-free MSE remains much larger than the Cramér-Rao bound.²

V. CONCLUSION

An interval error-based method has been developed to predict the mean-squared error performance of the full multinsnapshot and multifrequency MFP problem with possible model mismatch. The deterministic signal model has been considered. Unlike the stochastic model which considers that the source signals follow a Gaussian process, the deterministic model considers that the source signals are deterministic but unknown and makes no assumption on the source signal distribution. It is therefore appropriate for tonal components that may arise from submarines, ships and deployed tomographic sources or for marine mammals’ vocalizations [21]–[25]. Hence, the results provided in this article offer a good framework to analyze MFP performance in various configurations. Furthermore, these results could also be used for the matched-mode processing (MMP) problem [26]–[28] or for coherent MFP [14].

²Note that in the absence of mismatch, the asymptotic MSE as computed in (21) and (22) coincides with the CRB (not shown here due to lack of space).
REFERENCES


