Testing the Energy of Random Signals in a Known Subspace: an Optimal Invariant Approach
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Abstract—We consider the problem of testing whether the energy of a random signal projected onto a known subspace exceeds some specified value \( \tau \geq 0 \). The probability distribution of the signal is assumed to be unknown and this signal is observed in additive and independent white Gaussian noise with known variance. The proposed theoretical framework relies on the invariance of the problem and the resulting test is shown to be uniformly most powerful invariant in an extended sense suitable for random signals. This work extends Scharf and Friedlander's matched subspace detector.

Index Terms—Subspace Detection, UMPI Tests, Random Distortion Testing

I. INTRODUCTION

On the basis of noisy observations, matched subspace approaches are commonly used for deciding whether a signal, modeled by a set of unknown deterministic coordinates in a known subspace, is present or absent [1]–[3]. The derivation of such detectors is usually performed by studying the invariance classes for the problem at hand. Under certain conditions, subspace detectors belong to the family of uniformly most powerful invariant (UMPI) tests [1]. However, in case of model mismatch with signals that partly lie outside the considered subspace, these detectors are no longer UMPI.

This scenario occurs in many practical situations because of environmental fluctuations that distort the actual signal subspace compared to the assumed one. Therefore, testing whether a signal exactly lies within a given subspace may be too severe. A more flexible approach would be to allow for possible distortion by testing whether the energy of the observed signal in a subspace of interest exceeds some specified value. Such a test may have applications in radar or sonar when the propagation channel is unknown or when the null hypothesis does not reduce to noise only but includes the possible presence of signals of no interest.

As in [4]–[6], the present work is motivated by situations where the uncertainty on the signal’s nature is so important that the signal is assumed to be a random vector with unknown distribution. This is an additional particularity compared to matched subspace detectors that either consider deterministic signals or random signals evolving in a known class of probability density functions [7], [8].

Assuming a random signal of unknown distribution observed in additive and independent white Gaussian noise with known variance, we therefore address the problem of testing whether the energy of the signal projected onto a known subspace exceeds some specified value. Our main contribution is to show that a simple thresholding test on the energy of the observation projected onto the subspace of interest is optimal with respect to the invariance of the problem. This result is based on optimality properties recently established in [6].

This paper is organized as follows. The main definition and theoretical results associated with the invariance of the problem are presented in Section II. Numerical examples illustrating the performance of the proposed test are shown in Section III, followed by conclusions in Section IV.

Notation: All the random vectors and variables encountered henceforth are assumed to be defined on the same probability space \((\Omega, \mathcal{B}, \mathbb{P})\). As usual, we write (a-s) for almost surely. For any natural number \(d\), the set of all \(d\)-dimensional real random vectors defined on \((\Omega, \mathcal{B})\) is hereafter denoted by \(\mathcal{M}(\Omega, \mathbb{R}^d)\). The superscripts \(T\) denotes matrix transposition and \(\| \cdot \|\) designates the standard Euclidean norm in any given finite dimensional real vector space.

II. INVARiance AND OPTIMAL TEST

A. Problem statement and invariance

Let \(y, s, w\) be three elements of \(\mathcal{M}(\Omega, \mathbb{R}^N)\), where \(w \sim \mathcal{N}(0, \sigma^2 I_N)\) with known \(\sigma > 0\), \(s\) and \(w\) are independent and \(y = s + w\). Let \(\Psi\) denote an \(N \times n\) matrix that spans a rank-\(n\) subspace with \(n \leq N\) and \(\mathbf{P}_\Psi\) its associated projection matrix, i.e., \(\mathbf{P}_\Psi = \Psi \Psi^T\). The following elementary remarks will prove useful in the sequel. \(\Psi^T\Psi\) satisfies the following eigen-decomposition, \(\Psi^T\Psi = U \Delta U^T\), where \(\Delta\) is a \(n \times n\) diagonal matrix and \(U\) is \(n \times n\) orthonormal. Therefore, we have:

\[
\mathbf{P}_\Psi = U \Psi \Psi^T, \quad \mathbf{U}_\Psi = \Psi U \Delta^{-1/2}. \tag{1}
\]

Given a tolerance \(\tau \geq 0\), we address the problem of testing whether \(\|\mathbf{P}_\Psi s(\omega)\|^2 \geq \tau\) or not, when we are given \(y(\omega)\) for some \(\omega \in \Omega\) and when the probability distribution of \(s\) is unknown. The problem is summarized as follows:

\[
\begin{align*}
\text{Observation: } & y = s + w, \begin{cases} s \text{ and } w \text{ independent,} \\ s \in \mathcal{M}(\Omega, \mathbb{R}^N), \quad w \sim \mathcal{N}(0, \sigma^2 I_N), \end{cases} \\
\text{Null event: } & H_0 = \{\|\mathbf{P}_\Psi s\|^2 \leq \tau\}, \\
\text{Alternative event: } & H_1 = \{\|\mathbf{P}_\Psi s\|^2 > \tau\}. \tag{2}
\end{align*}
\]

Ideally, it is desirable to find a test satisfying some strong optimality criterion. Unfortunately, since \(s\) is random with unknown distribution, the methodology classically employed...
to build optimal tests [1] [9] in the presence of deterministic signals or random signals with known distribution do not apply. However, as discussed hereafter, problem (2) satisfies some properties of invariance. In such a situation, it is rather natural to restrict our attention to tests invariant to sets of transformations for which the problem is itself invariant and seek a test that meets some optimal criteria with respect to this invariance.

As far as invariance properties are concerned, problem (2) presents some analogy with what would be the same problem in the presence of a deterministic signal. In fact, it can be noticed that the invariance of our problem is strictly equivalent to the one encountered in the detection problem of a deterministic signal known to lie in a given subspace. The invariance can be formalized through the group of transforms in $\mathbb{R}^N$ [1, pp.146]:

$$\mathcal{G} = \{ g : g(u) = T(u+b), T = U_R RU^T_\Psi + P^\perp_\Psi, P^\perp_\Psi b = b \},$$

where $R$ is an $n \times n$ orthogonal matrix and $P^\perp_\Psi = I_N - P_\Psi$. The orbits of $\mathcal{G}$ are the cylinders $\mathcal{Y}_r = \{ u \in \mathbb{R}^N : \|P_\Psi u\|^2 = r \}$ with $r \geq 0$.

Given the invariance of our problem, it is desirable to find a test $T$ such that $T(\mathcal{G}(y)) = T(y)$. A sufficient condition to guarantee this equality is that $T$ be $\mathcal{G}$-invariant, that is, $T(\mathcal{G}(u)) = T(u)$ for all $u \in \mathbb{R}^N$ and all $g \in \mathcal{G}$. It can easily be checked that $u \in \mathbb{R}^N \mapsto \|P_\Psi u\|^2/\sigma^2 \in [0, \infty)$ is a maximal invariant of $\mathcal{G}$. Therefore, it follows from [9, Theorem 6.2.1] that the $\mathcal{G}$-invariant tests are functions of this maximal invariant.

### B. Deterministic case

Before addressing the general case where $s$ is random, let us first consider the situation where $s$ is deterministic, i.e., $s = s_\tau(a,s)$, where $s_\tau \in \mathbb{R}^N$.

In this case, since $w \sim N(0, \sigma^2 1_N)$, the maximal invariance statistic $\|P_\Psi w\|^2/\sigma^2 = \|P_\Psi w + P^\perp_\Psi w\|^2/\sigma^2$ follows a non-central $\chi^2$ probability density function (pdf) with $n$ degrees of freedom and noncentrality parameter $\rho$ equal to the ratio between the energy of $s_\tau$ in the subspace $\langle \Psi \rangle$ and the total noise power, i.e., $\rho = \|P_\Psi s_\tau\|^2/\sigma^2$. The family of such distributions is known to have monotone likelihood ratio with respect to its noncentrality parameter [10, Example A.1, pp. 468 - 469]. From Karlin-Rubin theorem [1], it then follows that the test defined for any $u \in \mathbb{R}^N$ by

$$\mathcal{T}_\lambda(u) = \begin{cases} 1 & \text{if } \|P_\Psi u\|^2 > \sigma^2 \lambda \\ 0 & \text{if } \|P_\Psi u\|^2 \leq \sigma^2 \lambda \end{cases}$$

is UMP invariant for problem (2) when $s = s_\tau(a,s)$. It remains to calculate $\lambda$ so as to guarantee a specified level.

Following standard definitions [9], we define the power function of a given test $T$ as

$$\beta_\lambda(T) = \mathbb{P}[T(s+\omega) = 1].$$

1. This invariance is the result of the following properties: $g(y) = g(s) + T u w$, with $g(a) \in M(\Omega, \mathbb{R}^N)$, $T u w \sim N(0, \sigma^2 1_N)$ and $\|P_\Psi g(s)\|^2 = \|P_\Psi s\|^2$.

2. $G$ introduces bias in $\langle \Psi^\perp \rangle$ and rotation in $\langle \Psi \rangle$ which can be represented as a cylinder, see illustration in [1, pp.131].

From (4), the size of $\mathcal{T}$ for problem (2) is defined as

$$\alpha(\mathcal{T}) = \sup_{s \in \mathbb{R}^N : \|P_\Psi s\|^2 < \gamma} \beta_\lambda(\mathcal{T}).$$

Given $\gamma \in (0, 1)$, $\mathcal{T}$ is said to have level (resp. size) $\gamma$ if $\alpha(\mathcal{T}) \leq \gamma$ (resp. $\alpha(\mathcal{T}) = \gamma$). Throughout the paper, $\mathcal{K}_\gamma$ denotes the class of tests with level $\gamma$.

As established in Appendix A, given any threshold $\lambda \in (0, \infty)$, the power function of $\mathcal{T}_\lambda$ depends on the actual signal energy-to-noise ratio $\rho$ and satisfies

$$\beta_\lambda(\mathcal{T}_\lambda) = Q_{n/2}\left(\sqrt{\rho \lambda}, \sqrt{\gamma}\right),$$

where $Q_\nu(a,b) \triangleq \frac{1}{\pi} \int_0^{\infty} t^\nu e^{-\frac{t^2 + a^2}{2}} I_{\nu-1}(at)dt$ is the generalized Marcum $Q$-function of real order $\nu > 0$ and $I_{\nu-1}$ is the modified Bessel function of the first kind of order $\nu-1$. The size of $\mathcal{T}_\lambda$ is then obtained as a consequence of the continuity and positive monotonicity of the Marcum functions with respect to their first parameter [11, Th. 1], i.e.,

$$\alpha(\mathcal{T}_\lambda) = Q_{n/2}\left(\sqrt{\tau/\sigma^2}, \sqrt{\gamma}\right).$$

### C. Random case

The standard methodology for deriving the UMPI test based on a maximal invariant statistic and monotony of the likelihood ratio cannot be applied if $s \in M(\Omega, \mathbb{R}^N)$ because the distribution of $s$ is simply unknown. To derive the UMPI test for (2) with a random signal, and actually show that this test is $\mathcal{T}_\lambda$, we hereafter rely on recent results concerning the optimality of thresholding tests on the energy of the observation [6].

For the sake of clarity, let us first state the definition of a $\mathcal{G}$-UMPI test in the case where $s$ is random with unknown distribution.

**Definition 1** Test $\mathcal{T}^*$ is said to be $\mathcal{G}$-UMPI with level $\gamma \in (0, 1)$ if

$$\sup_{s \in \mathcal{M}(\Omega, \mathbb{R}^N); P(H_0) \neq 0} \mathbb{P}\left[\mathcal{T}^*(y) = 1 \mid H_0\right] \leq \gamma$$

and $\mathbb{P}\left[\mathcal{T}^*(y) = 1 \mid H_1\right] \geq \mathbb{P}\left[\mathcal{T}(y) = 1 \mid H_1\right]$ for any $\mathcal{G}$-invariant test $\mathcal{T}$ in $\mathcal{K}_\gamma$ and any $s \in \mathcal{M}(\Omega, \mathbb{R}^N)$ such that $\mathbb{P}(H_1) \neq 0$.

As a preliminary result, note that

**Lemma 1** For any test $\mathcal{T}$ and any distribution of $s \in \mathcal{M}(\Omega, \mathbb{R}^N)$, we have:

$$\sup_{s \in \mathcal{M}(\Omega, \mathbb{R}^N); P(H_0) \neq 0} \mathbb{P}\left[\mathcal{T}(y) = 1 \mid H_0\right] = \alpha(\mathcal{T}),$$

where $\alpha(\mathcal{T})$ is the size given by (5) in the deterministic case.

**Proof:** See Appendix B.

For any random signals $s \in \mathcal{M}(\Omega, \mathbb{R}^N)$, the size of $\mathcal{T}_\lambda$ is therefore given by (7). We now state the properties of optimality satisfied by $\mathcal{T}_\lambda$ for any random signal $s \in \mathcal{M}(\Omega, \mathbb{R}^N)$.

**Proposition 1** Given any $\gamma \in (0, 1)$ and any $\tau \geq 0$, for $\lambda$ satisfying

$$Q_{n/2}\left(\sqrt{\tau/\sigma^2}, \sqrt{\gamma}\right) = \gamma,$$

$\mathcal{T}_\lambda \in \mathcal{K}_\gamma$ is $\mathcal{G}$-UMPI.

**Proof:** See Appendix C.
III. APPLICATION TO SIGNAL DETECTION IN SUBSPACE INTERFERENCE AND NOISE

To illustrate the interest of test \( T_\lambda \), we (re)consider the problem of detecting a known signal in subspace interference and noise with known level as presented in [2, Sec. V]. The original problem is to test

\[
\begin{cases}
H_0 : y = \Phi \xi + w, \\
H_1 : y = \mu z + \Phi \xi + w,
\end{cases}
\]

(10)

where the signal \( z \) is known and \( \mu \neq 0 \), the deterministic subspace interference \( \Phi \xi \) lies in a known subspace \( \Phi \) and the noise satisfies \( w \sim \mathcal{N}(0, \sigma^2 I_N) \) with \( \sigma^2 \) known. Given a specified level \( \gamma \), the resulting matched subspace detector (MSD) for this problem is \( \mathcal{G} \)-UMPI with size \( \eta \) and is expressed as [2, Eq. (5.18)]

\[
T^{\text{MSD}}_{\eta}(y) = \| P_{\Phi \perp} y \|^2 \frac{H_1}{H_0} \geq \sigma^2 \eta,
\]

(11)

where \( \eta \) satisfies \( \gamma = 1 - P[|\hat{\zeta}(0)| \leq \eta] = Q_{|\hat{\zeta}|}(0, \sqrt{\eta}) \). This test can be interpreted as an interference rejecting processor followed by a matched filter.

Some of the assumptions underlying (10) may be questionable in some practical situations. For instance, assuming that interferences have deterministic coordinates in a perfectly known subspace may be too optimistic. In fact, in applications such as sonar, systems experience interference generated by surrounding sources that share the same frequency channel. The nature of these competing sources is generally not known exactly and often random. The optimistic assumption on \( \Phi \xi \) translates in (11) by computing the threshold \( \eta \) on the basis that the energy of the interference in the rank-1 subspace \( \langle P_{\Phi \perp} z \rangle \) is null. To account for possible model mismatch we can recast (10) and formulate it in the framework of (2), that is

- **Observation:** \( y = \mu z + \zeta + w \)
- **Null event:** \( H_0 = \left\{ \| P_{\Phi \perp}(\mu z + \zeta) \|^2 \leq \tau \right\} \)
- **Alternative event:** \( H_1 = \left\{ \| P_{\Phi \perp}(\mu z + \zeta) \|^2 > \tau \right\} \)

(12)

where \( \zeta \) is some interfering signal. This interference \( \zeta \) is here assumed to be random and to lie within \( \Phi \), a subspace in the “neighborhood” of \( \Phi \). The mismatch between \( \Phi \) and \( \Phi \) is bounded through the tolerance \( \tau \).

Based on the results derived in the previous section, it turns out that the \( \mathcal{G} \)-UMPI test for (12) has the same structure as (11) but with a different threshold, i.e.,

\[
T_{\lambda}(y) = \| P_{\Phi \perp} y \|^2 \frac{H_1}{H_0} \geq \sigma^2 \lambda,
\]

(13)

where \( \lambda \) satisfies \( \gamma = Q_{1/2}(\sqrt{\tau/\sigma^2}, \sqrt{\lambda}) \) for a specified level \( \gamma \). Note that \( T_{\lambda} \) is a generalization of \( T^{\text{MSD}}_{\eta}(y) \) as (11) corresponds to (13) in the particular case where \( \tau = 0 \). In addition, an important result deriving from Prop. 1 is that Scharf and Friedlander’s test is not only optimal for deterministic interference but also for random coordinates \( \xi \) with unknown distribution in (10).

We straightforwardly derive from Lemma 1 and (7) that, whatever the distribution of \( \zeta \), the false alarm of test (13) satisfies

\[
P_{FA} \leq Q_{1/2}(\sqrt{\tau/\sigma^2}, \sqrt{\lambda}) = \gamma.
\]

(14)

Test \( T_{\lambda} \) is robust in that it avoids false detections due to model mismatch. If (11) is used instead of (13) in situations with interference subspace mismatch, the actual false alarm probability of (11) will be greater than the specified value \( \gamma \), no matter how small \( \tau > 0 \) is.

To illustrate this aspect, we now present some numerical results. We set \( \mu = 0 \) and consider a uniformly distributed random interference signal \( \zeta \) enclosed in a volume whose boundaries are given by a sphere of radius \( \sigma \sqrt{\varrho} \), where \( \varrho \) corresponds to the maximal interference-to-noise ratio, i.e., \( \| \zeta \|^2 / \sigma^2 \leq \varrho \), and two planes parallel to \( \langle P_{\Phi \perp} z \rangle \) such that \( \| P_{\Phi \perp} \zeta \|^2 \leq \tau \). The geometry of the simulation is illustrated in Figure 1. In practice, the choice of \( \tau \) can be based on rather empirical considerations as it is a bound on mismatch. For this example, we choose \( \tau \) as a given percentage of the maximal interference energy \( \sigma^2 \varrho \) and define the mismatch-to-maximal interference ratio as \( \nu = \Delta / (\sigma^2 \varrho) \). Figure 2 shows the actual false alarm probabilities for both test (13) and (11) as a function of the level \( \gamma \) and for different mismatch-to-maximal interference ratios. The maximal interference-to-noise ratio is set to 15 dB. As expected, simulations confirm that for any
\( \tau > 0 \) the matched subspace detector yields a false-alarm rate higher than specified. This is in contrast with test (13) that guarantees a false-alarm rate lower than the specified level, for any random interference signal. Due to lack of space, the receiver operating characteristics for detector (13) are not shown here but are available at https://sites.google.com/site/fxsanchez/publications/download/SPL14ROCcurves.pdf.

### IV. Conclusion

A simple thresholding test on the energy of the observation projected onto the subspace of interest has been shown to be UMPI for our problem. Contrary to what one might think, this is not the strongest statement of optimality that could be hoped for such a test. The recent results presented in [6] and the methodology adopted in this paper give us good reasons to conjecture that \( T_\lambda \) may satisfy the criterion of optimality known as the maximal constant conditional power (MCCP). This criterion is more general than the UMPI property [6, Th. 2]. In fact, for a specified level, a MCCP test has the greatest constant power on each orbit \( \Gamma_r \), with \( r > \tau \), in comparison to any other test with constant power on at least one orbit (but not necessarily on each of them). This statement remains to be proved and will be studied as a future work.

### Appendix A

#### Proof of (6)

We derive from (1) that

\[
\| P_\theta u \| = \| U_\theta^T u \| \text{ for any } u \in \mathbb{R}^N
\]  

(15)

and

\[
\| U_\theta^T w \sim N(0, \sigma^2 I_n). \]

(16)

Therefore, \( \beta_\lambda(T_\lambda) = \frac{\| U_\theta^T w + U_\theta^T w \|^2}{\sigma^2 \lambda} \) is the right-tail probability of a non-central \( \chi^2 \) distribution with \( n \) degrees of freedom and non-centrality parameter \( \rho \). This probability can thus be expressed as a Marcum function [12], i.e.

\[
\beta_\lambda(T_\lambda) = Q_{n/2}(\sqrt{\rho}, \sqrt{\lambda}).
\]

### Appendix B

#### Proof of Lemma 1

We begin with the following facts, which will also be used to establish Proposition 1. First, for any given test \( T \) defined for every \( u \in \mathbb{R}^N \), let test \( \overline{T} \) be defined for every \( v \in \mathbb{R}^N \) by setting:

\[
\overline{T}(v) = (T \circ U_\theta)(v) = T(U_\theta v).
\]

(17)

It follows from (1) that:

\[
T = \overline{T} \circ U_\theta^T.
\]

(18)

Henceforth, we put \( x = U_\theta^T w \). We then establish that:

\[
\alpha(T) = \sup_{\theta \in \mathbb{R}^n : \| \theta \| \leq \sqrt{\tau}} \mathbb{P}[\overline{T}(\theta + x) = 1].
\]

(19)

To this end, consider \( s \in \mathbb{R}^N \) such that \( \| P_\theta s \|^2 \leq \tau \). According to (18), \( T(s + w) = \overline{T}(U_\theta^T s + x) \) and

\[
\mathbb{P}[T(s + w) = 1] = \mathbb{P}[\overline{T}(U_\theta^T s + x) = 1].
\]

(20)

It then follows from (15) that \( \| U_\theta^T s \| \leq \sqrt{\tau} \), so that:

\[
\mathbb{P}[T(s + w) = 1] \leq \sup_{\theta \in \mathbb{R}^n : \| \theta \| \leq \sqrt{\tau}} \mathbb{P}[\overline{T}(\theta + x) = 1]
\]

and \( \alpha(T) \leq \sup_{\theta \in \mathbb{R}^n : \| \theta \| \leq \sqrt{\tau}} \mathbb{P}[\overline{T}(\theta + x) = 1] \).

Conversely, given any \( \theta \in \mathbb{R}^n \) such that \( \| \theta \| \leq \sqrt{\tau} \), \( s = U_\theta \theta \) is an element of \( \mathbb{R}^N \) such that \( \| P_\theta s \| = \| \theta \| \leq \sqrt{\tau} \). It follows from (5) and (20) that:

\[
\mathbb{P}[\overline{T}(\theta + x) = 1] = \mathbb{P}[T(s + w) = 1] \leq \alpha(T).
\]

Since \( \theta \in \mathbb{R}^n \) such that \( \| \theta \| \leq \sqrt{\tau} \) has arbitrarily been chosen, it follows that \( \sup_{\theta \in \mathbb{R}^n : \| \theta \| \leq \sqrt{\tau}} \mathbb{P}[\overline{T}(\theta + x) = 1] \leq \alpha(T) \), which concludes the proof of (19).

Let \( s \in M(\Omega, \mathbb{R}^N) \) with \( \mathbb{P}(H_0) \neq 0 \). From Bayes’s rule, (15) and (18), we obtain

\[
\mathbb{P}[T(s + w) = 1 | H_0] = \mathbb{P}[U_\theta^T s + x = 1 | \| U_\theta^T s \| \leq \sqrt{\tau}].
\]

Therefore,

\[
\mathbb{P}[T(s + w) = 1 | H_0] \leq \sup_{\theta \in \mathbb{M}(\Omega, \mathbb{R}^N): \| \theta \| \leq \sqrt{\tau}} \mathbb{P}[\overline{T}(\theta + x) = 1 | \| \theta \| \leq \sqrt{\tau}].
\]

According to [6, Lemma 4] and (19), we then have:

\[
\sup_{\theta \in \mathbb{M}(\Omega, \mathbb{R}^N): \| \theta \| \leq \sqrt{\tau}} \mathbb{P}[\overline{T}(\theta + x) = 1 | \| \theta \| \leq \sqrt{\tau}] = \sup_{\theta \in \mathbb{R}^n : \| \theta \| \leq \sqrt{\tau}} \mathbb{P}[\overline{T}(\theta + x) = 1] = \alpha(T).
\]

Thus, \( \sup_{s \in \mathbb{M}(\Omega, \mathbb{R}^N): \| \theta \| \leq \sqrt{\tau}} \mathbb{P}[T(s + w) = 1 | H_0] \leq \alpha(T) \).

Conversely, if \( s = s \) (a.s) with \( \| P_\theta s \|^2 \leq \tau \), we have \( \mathbb{P}[T(s + w) = 1] \leq \mathbb{P}[T(s + w) = 1 | H_0] \leq \alpha(T) \).

Since \( \alpha(T) \leq \sup_{s \in \mathbb{M}(\Omega, \mathbb{R}^N): \| \theta \| \leq \sqrt{\tau}} \mathbb{P}[T(s + w) = 1 | H_0] \), which concludes the proof.

### Appendix C

#### Proof of Proposition 1

Consider \( T_\lambda \) where \( \lambda \) is the unique solution to (9) and \( T \in \mathcal{K}_\gamma \). It follows from (7) and Lemma 1 that \( T_\lambda \) (resp. \( T \)) has size (resp. level) \( \gamma \) for problem (2). Let \( s \in \mathbb{R}^n \) and \( \theta \in \Omega \). According to (15), \( T(s + w) = \overline{T}(U_\theta^T s + x) \) and \( \| P_\theta s \|^2 \leq \tau \). Suppose \( s \in \mathbb{R}^n \) is a maximal invariant of \( \mathcal{G} \). It follows that \( T \) is \( \mathcal{O}_\gamma \)-invariant for any \( \mathcal{G} \)-invariant \( T \). In particular, \( T_\lambda \) is \( \mathcal{O}_\gamma \)-invariant as well. Suppose that \( s \) is such that \( \| P_\theta s \|^2 \leq \tau \). We then have \( \mathbb{P}[\| \theta \| > \sqrt{\tau}] \neq 0 \) thanks to (15). Since a straightforward application of (17) leads to:

\[
\overline{T}(u) = \begin{cases} 1 & \text{if } \| u \|^2 > \sigma^2 \lambda \\ 0 & \text{otherwise} \end{cases}, \text{ for } u \in \mathbb{R}^n
\]

we derive from [6, Theorem 2 (iii) & Eq. (9)] that:

\[
\mathbb{P}[\overline{T}(\theta + x) = 1 | \| \theta \| > \sqrt{\tau}] \geq \mathbb{P}[\overline{T}(\theta + x) = 1 | \| \theta \| > \sqrt{\tau}].
\]

Thence the result since an easy computation based on Bayes’s rule shows that

\[
\mathbb{P}[T(y) = 1 | H_1] = \mathbb{P}[T(\theta + x) = 1 | \| \theta \| > \sqrt{\tau}]
\]

for any test \( T \in \mathbb{R}^N \).
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